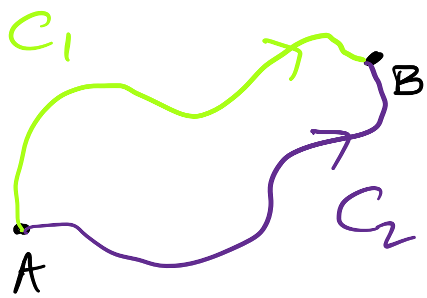
Conservative vector fields recap



For a general vector field F,

$$\int_{C_1} F \cdot dr \neq \int_{C_2} F \cdot dr$$

$\boxed{\text{BUT}}$ if $F = \nabla \varphi$ (i.e. F is <u>conservative</u>

then $$\int_{C_1} F \cdot dr = \int_{C_2} F \cdot dr = \varphi(B) - \varphi(A)$$

This is called path-independence.

If F is conservative, and we find a potential φ, $F = \nabla \varphi$, then evaluating line integrals becomes EASY!

THEOREM 17.6 Line Integrals on Closed Curves
Let R be an open connected region in $\mathbb{R}^2$ or $\mathbb{R}^3$. Then $\mathbf{F}$ is a conservative vector field on R if and only if $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ on all simple closed piecewise-smooth oriented curves C in R.

$$\text{Path independence} \quad \overbrace{\Leftrightarrow}^{\text{Theorems 17.4 and 17.5}} \quad \mathbf{F} \text{ is conservative } (\nabla\varphi = \mathbf{F}) \quad \overbrace{\Leftrightarrow}^{\text{Theorem 17.6}} \quad \oint_C \mathbf{F} \cdot d\mathbf{r} = \mathbf{0}.$$

THEOREM 17.3 Test for Conservative Vector Fields

Let $\mathbf{F} = \langle f, g, h \rangle$ be a vector field defined on a connected and simply connected region D of $\mathbb{R}^3$, where f, g, and h have continuous first partial derivatives on D. Then $\mathbf{F}$ is a conservative vector field on D (there is a potential function φ such that $\mathbf{F} = \nabla\varphi$) if and only if

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}, \qquad \frac{\partial f}{\partial z} = \frac{\partial h}{\partial x}, \quad \text{and} \quad \frac{\partial g}{\partial z} = \frac{\partial h}{\partial y}.$$

For vector fields in $\mathbb{R}^2$, we have the single condition $\dfrac{\partial f}{\partial y} = \dfrac{\partial g}{\partial x}$.

EXAMPLE 2 Determine whether or not the given vector field is conservative.

(a) $\mathbf{F}(x, y) = (x - y)\,\mathbf{i} + (x - 2)\,\mathbf{j}$

(b) $\mathbf{F}(x, y) = (3 + 2xy)\,\mathbf{i} + (x^2 - 3y^2)\,\mathbf{j}$

$\mathbf{F} = \langle 2xy - z^2, x^2 + 2z, 2y - 2xz \rangle$

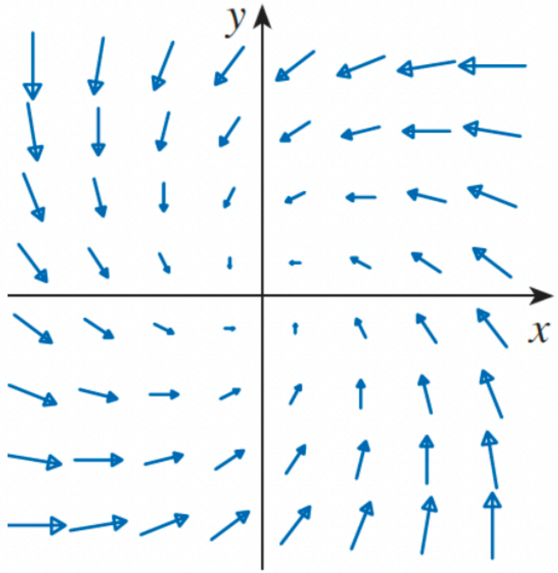
Finding the potential for a conservative vector field by integration

Ex: find a potential for each of the following conservative vector fields:

$\mathbf{F}(x, y) = (3 + 2xy)\,\mathbf{i} + (x^2 - 3y^2)\,\mathbf{j}$

$\mathbf{F} = \langle 2xy - z^2, x^2 + 2z, 2y - 2xz \rangle$

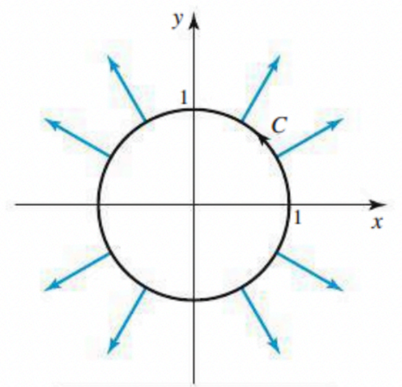
Is the vector field shown in the figure conservative?



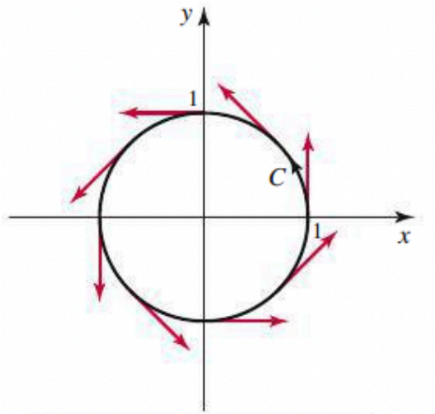
Circulation and Flux - two other words associated with line integrals of vector fields over closed curves.

DEFINITION Circulation

Let $\mathbf{F}$ be a continuous vector field on a region D of $\mathbb{R}^3$, and let C be a closed smooth oriented curve in D. The **circulation** of $\mathbf{F}$ on C is $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$, where $\mathbf{T}$ is the unit vector tangent to C consistent with the orientation.



On the unit circle, $\mathbf{F} = \langle x, y \rangle$ is orthogonal to C and has zero circulation on C.



On the unit circle, $\mathbf{F} = \langle -y, x \rangle$ is tangent to C and has positive circulation on C.

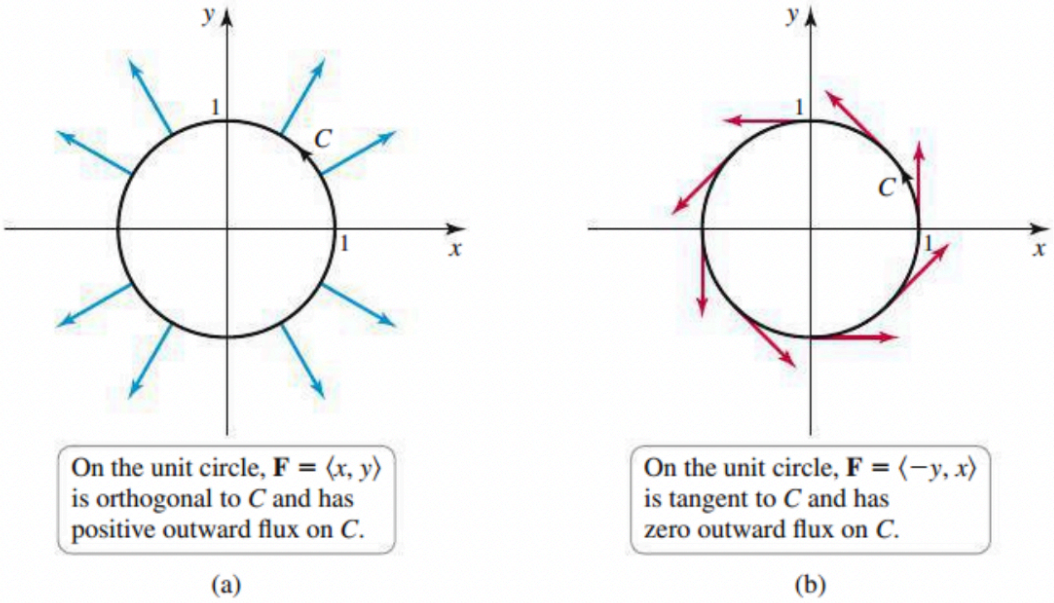
DEFINITION Flux

Let $\mathbf{F} = \langle f, g \rangle$ be a continuous vector field on a region R of $\mathbb{R}^2$. Let C: $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, for $a \leq t \leq b$, be a smooth oriented curve in R that does not intersect itself. The **flux** of the vector field $\mathbf{F}$ across C is

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_a^b (f(t)y'(t) - g(t)x'(t))\, dt,$$

where $\mathbf{n} = \mathbf{T} \times \mathbf{k}$ is the unit normal vector and $\mathbf{T}$ is the unit tangent vector consistent with the orientation. If C is a closed curve with counterclockwise orientation, $\mathbf{n}$ is the outward normal vector, and the flux integral gives the **outward flux** across C.

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_C f\, dy - g\, dx.$$


y
x
1
1
C
On the unit circle, $\mathbf{F} = \langle x, y \rangle$ is orthogonal to C and has positive outward flux on C.
(a)
y
x
1
1
C
On the unit circle, $\mathbf{F} = \langle -y, x \rangle$ is tangent to C and has zero outward flux on C.
(b)

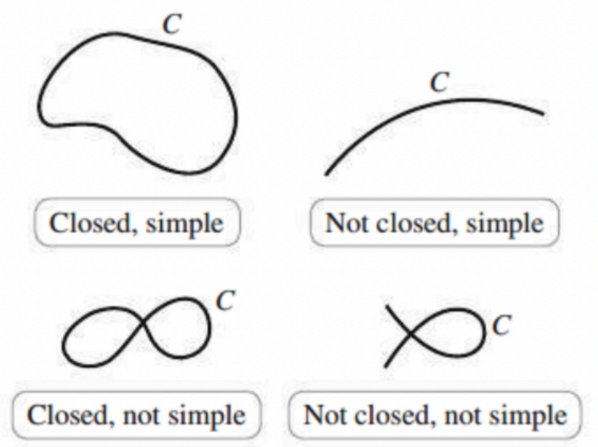
17.4 Green's theorem

first of three theorems relating integrals over the boundary with integrals over the interior

Types of Curves and Regions

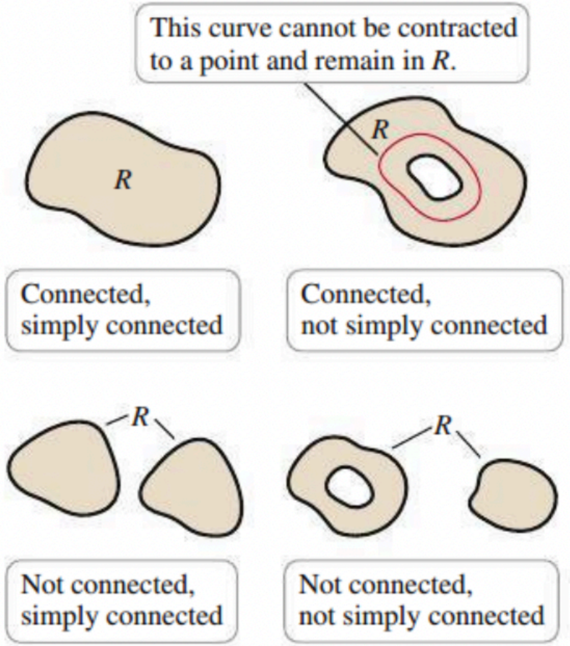
Many of the results in the remainder of this text rely on special properties of regions and curves. It's best to collect these definitions in one place for easy reference.

> **DEFINITION Simple and Closed Curves**
>
> Suppose a curve C (in $\mathbb{R}^2$ or $\mathbb{R}^3$) is described parametrically by $\mathbf{r}(t)$, where $a \le t \le b$. Then C is a **simple curve** if $\mathbf{r}(t_1) \ne \mathbf{r}(t_2)$ for all t_1 and t_2, with $a < t_1 < t_2 < b$; that is, C never intersects itself between its endpoints. The curve C is **closed** if $\mathbf{r}(a) = \mathbf{r}(b)$; that is, the initial and terminal points of C are the same (Figure 17.28).



DEFINITION Connected and Simply Connected Regions

An open region R in $\mathbb{R}^2$ (or D in $\mathbb{R}^3$) is **connected** if it is possible to connect any two points of R by a continuous curve lying in R. An open region R is **simply connected** if every closed simple curve in R can be deformed and contracted to a point in R (Figure 17.29).

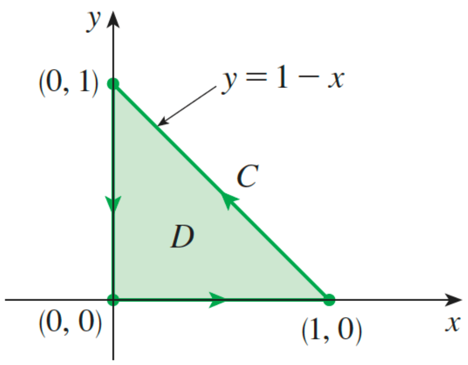


THEOREM 17.7 Green's Theorem—Circulation Form

Let C be a simple closed piecewise-smooth curve, oriented counterclockwise, that encloses a connected and simply connected region R in the plane. Assume $\mathbf{F} = \langle f, g \rangle$, where f and g have continuous first partial derivatives in R. Then

$$\underbrace{\oint_C \mathbf{F} \cdot d\mathbf{r}}_{\text{circulation}} = \underbrace{\oint_C f\,dx + g\,dy}_{\text{circulation}} = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA.$$

EXAMPLE 1 Evaluate $\int_C x^4\,dx + xy\,dy$, where C is the triangular curve consisting of the line segments from (0, 0) to (1, 0), from (1, 0) to (0, 1), and from (0, 1) to (0, 0).



EXAMPLE 2 Evaluate $\oint_C (3y - e^{\sin x})\,dx + \left(7x + \sqrt{y^4 + 1}\right) dy$, where C is the circle $x^2 + y^2 = 9$.

THEOREM 17.8 Area of a Plane Region by Line Integrals
Under the conditions of Green's Theorem, the area of a region R enclosed by a curve C is

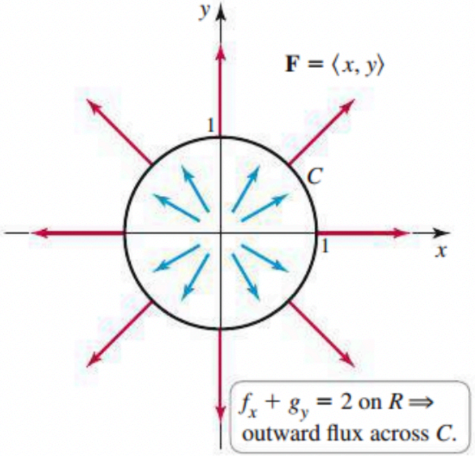
$$\oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C (x\,dy - y\,dx).$$

EXAMPLE 2 Area of an ellipse Find the area of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

THEOREM 17.9 Green's Theorem—Flux Form
Let C be a simple closed piecewise-smooth curve, oriented counterclockwise, that encloses a connected and simply connected region R in the plane. Assume $\mathbf{F} = \langle f, g \rangle$, where f and g have continuous first partial derivatives in R. Then

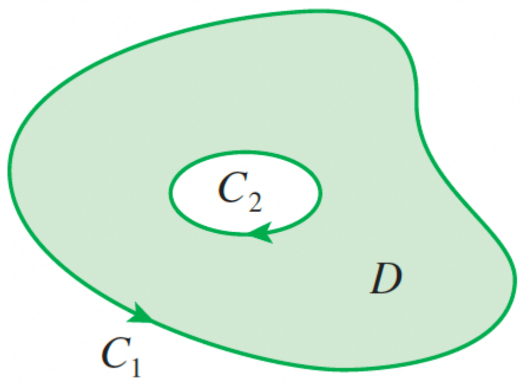
$$\underbrace{\oint_C \mathbf{F} \cdot \mathbf{n}\, ds}_{\text{outward flux}} = \underbrace{\oint_C f\, dy - g\, dx}_{\text{outward flux}} = \iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right) dA,$$

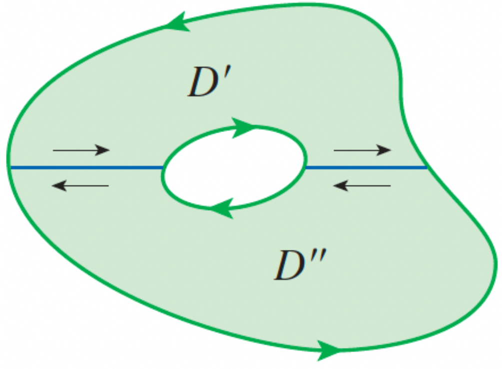
where $\mathbf{n}$ is the outward unit normal vector on the curve.



EXAMPLE 3 Outward flux of a radial field Use Green's Theorem to compute the outward flux of the radial field $\mathbf{F} = \langle x, y \rangle$ across the unit circle $C = \{(x, y): x^2 + y^2 = 1\}$

Extending Green’s theorem to regions with holes





EXAMPLE 5 If $\mathbf{F}(x, y) = (-y\,\mathbf{i} + x\,\mathbf{j})/(x^2 + y^2)$, show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$ for every positively oriented simple closed path that encloses the origin.

$\oint_C xy\,dx + x^2y^3\,dy$,
C is the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 2)$