MA 442 - Practice final exam

Name: __________________________________ **BUID:** ______________________

There are six problems, you must solve **all** of them. Each problem is worth **10 points**. You have 120 minutes to complete this exam.

This exam is closed-book, but you are allowed a single cheat sheet. No electronic devices are permitted.

Question 1. Provide precise definitions of the following terms.

(a) A **basis** of a vector space.

(b) The **kernel** of a linear transformation.

(c) The **determinant** of an $n \times n$ matrix (as a multilinear alternating function).

(d) An **eigenvalue** and **eigenvector** of a linear transformation.

(e) A T-**invariant** subspace of a vector space.

Question 2. Suppose the linear transformation $T\colon V \to V$ satisfies $T^3 = 0$.

(a) Show that $\mathrm{im}(T^2) \subseteq \ker(T)$.

(b) Show that

$$\mathrm{rank}(T) \leq \frac{2}{3} \dim V.$$

Question 3. This question has two parts.

(a) Let V, W be vector spaces. Show that $\mathsf{T}\colon \mathsf{V} \longrightarrow \mathsf{W}$ is injective if and only if there exists $\mathsf{S}\colon \mathsf{W} \longrightarrow \mathsf{V}$ such that

$$\mathsf{S} \circ \mathsf{T} = \mathbb{1}_{\mathsf{V}}.$$

(b) Suppose A, B are square matrices such that $ABA = A$. Show that

$$\operatorname{rank}(A) = \operatorname{rank}(AB).$$

Question 4. Define a linear transformation

$$\mathsf{T}\colon \mathsf{P}_2 \longrightarrow \mathsf{P}_3$$

defined by

$$\mathsf{T}(f)(x) = xf'(x) - 2f(x) + \left(\int_0^1 f(t)\,dt\right)(1+x).$$

(a) Compute the matrix of T with respect to the bases $\{1, x, x^2\}$ of P_2 and $\{1, x, x^2, x^3\}$ of P_3.

(b) Find a basis for $\ker(\mathsf{T})$ and compute $\operatorname{rank}(\mathsf{T})$.

(c) Determine whether T is injective, surjective, or neither. Justify your answer.

(d) Does there exist a polynomial f such that $\mathsf{T}(f) = x^3$?

Question 5. Let A, B be diagonalizable $n \times n$ matrices over $\mathbb{R}$ such that

$$AB = BA.$$

(a) Show that for any eigenvalue λ of A, the eigenspace

$$E_\lambda = \ker(A - \lambda I)$$

is invariant under B.

(b) Deduce that A and B are simultaneously diagonalizable.

Question 6. Let A be a 3×3 matrix such that

$$\det(A) = 20, \qquad \operatorname{tr}(A) = 9.$$

Suppose moreover that 2 is an eigenvalue of A, and that the eigenspace

$$E_2 = \ker(A - 2I)$$

has dimension 2.

(a) Determine the characteristic polynomial of A.

(b) Is A diagonalizable?