MA 442 - Practice final exam

Name: ______________________________ **BUID:** ____________________

There are six problems, you must solve **all** of them. Each problem is worth **10 points**. You have 120 minutes to complete this exam.

This exam is closed-book, but you are allowed a single cheat sheet. No electronic devices are permitted.

Question 1. Provide precise definitions of the following terms.

(a) A **basis** of a vector space.

(b) The **kernel** of a linear transformation.

(c) The **determinant** of an $n \times n$ matrix (as a multilinear alternating function).

(d) An **eigenvalue** and **eigenvector** of a linear transformation.

(e) A T-**invariant** subspace of a vector space.

Solution. (a) A basis of a vector space V is a list of vectors $v_1, \ldots, v_n \in \mathsf{V}$ which is linearly independent and spans V. Equivalently, every vector $v \in \mathsf{V}$ can be written uniquely as $v = a_1 v_1 + \cdots + a_n v_n$.

(b) If $\mathsf{T}\colon \mathsf{V} \to \mathsf{W}$ is a linear transformation, then the kernel of T is $\ker(\mathsf{T}) = \{v \in \mathsf{V} : \mathsf{T}(v) = 0\}$.

(c) The columns of a matrix A can be organized into an n-tupe of vectors

$$A = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix}. \tag{1}$$

The determinant of an $n \times n$ matrix A is the unique function of the columns

$$\det\colon \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}, \quad (\mathbf{v}_1, \ldots, \mathbf{v}_n) \mapsto \det(\mathbf{v}_1, \ldots, \mathbf{v}_n) = \det A \tag{2}$$

which is multilinear, alternating, and satisfies $\det(e_1, \ldots, e_n) = 1$, where $e_1, \ldots, e_n$ are the standard basis vectors of $\mathbb{R}^n$.

(d) Let $\mathsf{T}\colon \mathsf{V} \to \mathsf{V}$ be linear. A scalar λ is an eigenvalue of T if there exists a nonzero vector $v \in \mathsf{V}$ such that $\mathsf{T}(v) = \lambda v$. Such a nonzero vector v is called an eigenvector of T with eigenvalue λ.

(e) Let $\mathsf{T}\colon \mathsf{V} \to \mathsf{V}$ be linear. A subspace $\mathsf{U} \subseteq \mathsf{V}$ is called T-invariant if $\mathsf{T}(\mathsf{U}) \subseteq \mathsf{U}$. Equivalently, whenever $u \in \mathsf{U}$, we also have $\mathsf{T}(u) \in \mathsf{U}$.

Question 2. Suppose the linear transformation $\mathsf{T}\colon \mathsf{V} \to \mathsf{V}$ satisfies $\mathsf{T}^3 = 0$.

(a) Show that $\operatorname{im}(\mathsf{T}^2) \subseteq \ker(\mathsf{T})$.

(b) Show that

$$\operatorname{rank}(\mathsf{T}) \leq \frac{2}{3} \dim \mathsf{V}.$$

Solution. (a) Let $v \in \operatorname{im}(\mathsf{T}^2)$. Then $v = \mathsf{T}^2(w)$ for some $w \in \mathsf{V}$. Therefore $\mathsf{T}(v) = \mathsf{T}(\mathsf{T}^2(w)) = \mathsf{T}^3(w) = 0$. Thus $v \in \ker(\mathsf{T})$, so $\operatorname{im}(\mathsf{T}^2) \subseteq \ker(\mathsf{T})$.

(b) Let $n = \dim \mathsf{V}$. By rank-nullity, $n = \operatorname{rank}(\mathsf{T}) + \dim\ker(\mathsf{T})$. We will show that $\operatorname{rank}(\mathsf{T}) \leq 2\dim\ker(\mathsf{T})$.

Consider the restriction $\mathsf{T}|_{\operatorname{im}(\mathsf{T})}\colon \operatorname{im}(\mathsf{T}) \to \operatorname{im}(\mathsf{T})$. Its image is $\operatorname{im}(\mathsf{T}^2)$, and its kernel is $\operatorname{im}(\mathsf{T}) \cap \ker(\mathsf{T})$. By rank-nullity applied to this restriction,

$$\dim\operatorname{im}(\mathsf{T}) = \dim\big(\operatorname{im}(\mathsf{T}) \cap \ker(\mathsf{T})\big) + \dim\operatorname{im}(\mathsf{T}^2).$$

Now $\dim(\operatorname{im}(\mathsf{T}) \cap \ker(\mathsf{T})) \leq \dim\ker(\mathsf{T})$, and by part (a), $\operatorname{im}(\mathsf{T}^2) \subseteq \ker(\mathsf{T})$, so $\dim\operatorname{im}(\mathsf{T}^2) \leq \dim\ker(\mathsf{T})$. Hence $\operatorname{rank}(\mathsf{T}) = \dim\operatorname{im}(\mathsf{T}) \leq 2\dim\ker(\mathsf{T})$.

Thus $\operatorname{rank}(\mathsf{T}) \leq 2(n - \operatorname{rank}(\mathsf{T}))$. So $3\operatorname{rank}(\mathsf{T}) \leq 2n$, and therefore $\operatorname{rank}(\mathsf{T}) \leq \frac{2}{3}\dim\mathsf{V}$.

Question 3. This question has two parts.

(a) Let V, W be vector spaces. Show that $\mathsf{T}\colon \mathsf{V} \longrightarrow \mathsf{W}$ is injective if and only if there exists $\mathsf{S}\colon \mathsf{W} \longrightarrow \mathsf{V}$ such that

$$\mathsf{S} \circ \mathsf{T} = \mathbb{1}_{\mathsf{V}}.$$

(b) Suppose A, B are square matrices such that $ABA = A$. Show that

$$\operatorname{rank}(A) = \operatorname{rank}(AB).$$

Solution. (a) First suppose there exists $\mathsf{S}\colon \mathsf{W} \longrightarrow \mathsf{V}$ such that $\mathsf{S} \circ \mathsf{T} = \mathbb{1}_{\mathsf{V}}$. We show that T is injective. If $\mathsf{T}(v) = 0$, then applying S gives $\mathsf{S}(\mathsf{T}(v)) = \mathsf{S}(0) = 0$. But $\mathsf{S}(\mathsf{T}(v)) = v$, so $v = 0$. Hence $\ker(\mathsf{T}) = \{0\}$, so T is injective.

Conversely, suppose T is injective. Then $\mathsf{T}\colon \mathsf{V} \longrightarrow \operatorname{im}(\mathsf{T})$ is an isomorphism. Let $\mathsf{S}_0\colon \operatorname{im}(\mathsf{T}) \longrightarrow \mathsf{V}$ be its inverse. Choose a basis of $\operatorname{im}(\mathsf{T})$ and extend it to a basis of W. Define $\mathsf{S}\colon \mathsf{W} \longrightarrow \mathsf{V}$ by setting $\mathsf{S} = \mathsf{S}_0$ on $\operatorname{im}(\mathsf{T})$ and setting $\mathsf{S} = 0$ on the extra basis vectors. Then, for every $v \in \mathsf{V}$, we have $\mathsf{S}(\mathsf{T}(v)) = v$. Thus $\mathsf{S} \circ \mathsf{T} = \mathbb{1}_{\mathsf{V}}$.

(b) We always have $\operatorname{im}(AB) \subseteq \operatorname{im}(A)$. Therefore $\operatorname{rank}(AB) \leq \operatorname{rank}(A)$.

On the other hand, since $ABA = A$, we have $A = (AB)A$. Thus $\operatorname{im}(A) = \operatorname{im}((AB)A) \subseteq \operatorname{im}(AB)$. Therefore $\operatorname{rank}(A) \leq \operatorname{rank}(AB)$.

Combining the two inequalities gives $\operatorname{rank}(A) = \operatorname{rank}(AB)$.

Question 4. Define a linear transformation

$$T\colon P_2 \longrightarrow P_3$$

defined by

$$T(f)(x) = xf'(x) - 2f(x) + \left(\int_0^1 f(t)\,dt\right)(1+x).$$

(a) Compute the matrix of T with respect to the bases $\beta = \{1, x, x^2\}$ of P_2 and $\gamma = \{1, x, x^2, x^3\}$ of P_3.

(b) Find a basis for $\ker(T)$ and compute $\operatorname{rank}(T)$.

(c) Determine whether T is injective, surjective, or neither. Justify your answer.

(d) Does there exist a polynomial f such that $T(f) = x^3$?

Solution. Let $f(x) = a + bx + cx^2$. Then $f'(x) = b + 2cx$, so $xf'(x) = bx + 2cx^2$. Also $\int_0^1 f(t)\,dt = a + \frac{b}{2} + \frac{c}{3}$. Therefore

$$\begin{aligned} T(f)(x) &= xf'(x) - 2f(x) + \left(a + \frac{b}{2} + \frac{c}{3}\right)(1+x) \\ &= (bx + 2cx^2) - 2(a + bx + cx^2) + \left(a + \frac{b}{2} + \frac{c}{3}\right)(1+x) \\ &= \left(-a + \frac{b}{2} + \frac{c}{3}\right) + \left(a - \frac{b}{2} + \frac{c}{3}\right)x. \end{aligned}$$

(a) We compute T on the basis vectors: $T(1) = -1 + x$, $T(x) = \frac{1}{2} - \frac{1}{2}x$, and $T(x^2) = \frac{1}{3} + \frac{1}{3}x$. Therefore the matrix of T with respect to the bases $\beta = \{1, x, x^2\}$ and $\gamma = \{1, x, x^2, x^3\}$ is

$$[T]_\beta^\gamma = \begin{bmatrix} -1 & \frac{1}{2} & \frac{1}{3} \\ 1 & -\frac{1}{2} & \frac{1}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(b) The kernel consists of all $f(x) = a + bx + cx^2$ such that $-a + \frac{b}{2} + \frac{c}{3} = 0$ and $a - \frac{b}{2} + \frac{c}{3} = 0$. Adding these equations gives $\frac{2c}{3} = 0$, so $c = 0$. Then $-a + \frac{b}{2} = 0$, so $a = \frac{b}{2}$.

Thus $f(x) = \frac{b}{2} + bx = \frac{b}{2}(1+2x)$. Hence a basis for the kernel is $\{1+2x\}$. Since $\dim P_2 = 3$, rank-nullity gives $\operatorname{rank}(T) = 3 - \dim\ker(T) = 3 - 1 = 2$.

(c) The map T is not injective because $\ker(T) = \operatorname{span}\{1 + 2x\}$ is nonzero. It is also not surjective because $\operatorname{rank}(T) = 2 < 4 = \dim P_3$. Therefore T is neither injective nor surjective.

(d) No. For every $f \in P_2$, the polynomial $T(f)$ has no x^2 or x^3 term. Thus x^3 is not in the image of T. Therefore there does not exist a polynomial $f \in P_2$ such that $T(f) = x^3$.

Question 5. Let A, B be diagonalizable $n \times n$ matrices over $\mathbb{R}$ such that

$$AB = BA.$$

(a) Show that for any eigenvalue λ of A, the eigenspace

$$\mathsf{E}_\lambda = \ker(A - \lambda I)$$

is invariant under B.

(b) Deduce that A and B are simultaneously diagonalizable.

Solution. (a) Let $v \in \mathsf{E}_\lambda$. Then $Av = \lambda v$. We want to show that $Bv \in \mathsf{E}_\lambda$. Since $AB = BA$, we have $A(Bv) = B(Av)$. But $Av = \lambda v$, so $A(Bv) = B(\lambda v) = \lambda Bv$. Thus $Bv \in \mathsf{E}_\lambda$. Therefore E_λ is B-invariant.

(b) Since A is diagonalizable, every vector in $\mathbb{R}^n$ can be written, uniquely, as a sum of eigenvectors of A. In other words, there is a direct sum decomposition

$$\mathbb{R}^n = \oplus_\lambda \mathsf{E}_\lambda \tag{3}$$

where the sum is over distinct eigenvalues of A.

By part (a), each E_λ is invariant under B. Since B is diagonalizable on $\mathbb{R}^n$, its restriction $B|_{\mathsf{E}_\lambda} : \mathsf{E}_\lambda \longrightarrow E_\lambda$ is also diagonalizable (this was a problem on, or at least related to, the homework. If you have not proved this yet, please do so.) Therefore each eigenspace E_λ has a basis consisting of eigenvectors of B.

Taking all of these bases together gives a basis of $\mathbb{R}^n$ consisting of vectors which are eigenvectors for A and also eigenvectors for B. Therefore A and B are simultaneously diagonalizable.

Question 6. Let A be a 3×3 matrix such that

$$\det(A) = 20, \qquad \operatorname{tr}(A) = 9.$$

Suppose moreover that 2 is an eigenvalue of A, and that the eigenspace

$$\mathsf{E}_2 = \ker(A - 2I)$$

has dimension 2.

(a) Determine the characteristic polynomial of A.

(b) Is A diagonalizable?

Solution. (a) Since $\dim \mathsf{E}_2 = 2$, the eigenvalue 2 has geometric multiplicity 2. Therefore its algebraic multiplicity $m_\lambda \geq 2$. Since A is a 3×3 matrix, the eigenvalues of A are $2, 2, \lambda$ for some scalar λ.

Using the determinant (the determinant is the product of the eigenvalues), $\det(A) = 2 \cdot 2 \cdot \lambda = 4\lambda$. Since $\det(A) = 20$, we get $4\lambda = 20$, so $\lambda = 5$. Thus the eigenvalues are $2, 2, 5$.

The characteristic polynomial is therefore

$$\chi_A(t) = -(t-2)^2(t-5) = -t^3 + 9t^2 - 24t + 20.$$

This is also consistent with $\operatorname{tr}(A) = 2 + 2 + 5 = 9$, though we did not explicitly use this!

(b) Yes, A is diagonalizable. The eigenspace for $\lambda = 2$ has dimension 2 by assumption. Since 5 is a distinct eigenvalue, its eigenspace has dimension at least 1. Therefore we have at least $2 + 1 = 3$ linearly independent eigenvectors. Since A is a 3×3 matrix, these form a basis of $\mathbb{R}^3$. Hence A is diagonalizable.