MA 442 - Final exam practice

Name: ____________________________________ **BUID:** ________________________

There are six problems, you must solve **five** of them. Each problem is worth **10 points**. You have 90 minutes to complete this exam.

This exam is closed-book, but you are allowed a single cheat sheet.

Question 1. For each statement below, circle whether it is **True** or **False**.

Statement	**T**	**F**
(a) If A is an $n \times n$ matrix with $\det(A) = 0$, then A has an eigenvalue 0.	☐	☐
(b) If $AB = 0$ for square matrices A, B, then either $A = 0$ or $B = 0$.	☐	☐
(c) Every linearly independent set in a finite-dimensional vector space can be extended to a basis.	☐	☐
(d) If $\mathsf{T}\colon \mathsf{V} \to \mathsf{V}$ has $\ker(\mathsf{T}) = \{0\}$, then T is invertible.	☐	☐
(f) If A is diagonalizable, then A^2 is diagonalizable.	☐	☐
(g) If A^2 is diagonalizable, then A is diagonalizable.	☐	☐
(h) If A, B are diagonalizable and $AB = BA$, then they are simultaneously diagonalizable.	☐	☐
(i) The determinant is a linear function of the columns of a matrix.	☐	☐
(j) If λ is an eigenvalue of T, then $\mathsf{T} - \lambda I$ is not invertible.	☐	☐
(k) Every $n \times n$ real matrix has n real eigenvalues.	☐	☐
(l) If A is invertible, then $\det(A^{-1}) = (\det A)^{-1}$.	☐	☐
(o) If A is upper triangular, then its eigenvalues are the diagonal entries.	☐	☐
(p) Every square matrix satisfies its characteristic polynomial.	☐	☐
(q) If A satisfies $p(A) = 0$, then $p(t)$ must be a multiple of $\chi_A(t)$.	☐	☐
(r) If $\mathsf{U}, \mathsf{W} \subseteq \mathsf{V}$, then $\dim(\mathsf{U} + \mathsf{W}) = \dim(\mathsf{U}) + \dim(\mathsf{W})$.	☐	☐
(s) If $\mathsf{T}^2 = 0$, then $\operatorname{im}(\mathsf{T}) \subseteq \ker(\mathsf{T})$.	☐	☐
(t) Any two bases of a finite-dimensional vector space have the same size.	☐	☐

Solution. (a) **True.** Since $\det(A) = 0$, the matrix A is not invertible. Therefore $A - 0I = A$ is not invertible, so 0 is an eigenvalue of A.

(b) **False.** For example, let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then $AB = 0$, but neither A nor B is zero.

(c) **True.** This is the basis extension theorem.

(d) **True.** Since $\mathsf{T}\colon \mathsf{V} \to \mathsf{V}$ and V is finite-dimensional, $\ker(\mathsf{T}) = \{0\}$ implies that T is injective. By rank-nullity, T is also surjective, hence invertible.

(f) **True.** If A is diagonalizable, then $A = PDP^{-1}$ for some diagonal matrix D. Then $A^2 = PD^2P^{-1}$, and D^2 is diagonal. Hence A^2 is diagonalizable.

(g) **False.** For example, let
$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$
Then $A^2 = 0$, which is diagonalizable, but A is not diagonalizable.

(h) **True.** Commuting diagonalizable matrices are simultaneously diagonalizable.

(i) **True.** More precisely, the determinant is multilinear in the columns, meaning it is linear in each column separately.

(j) **True.** If λ is an eigenvalue of T, then there exists $v \neq 0$ such that $\mathsf{T}(v) = \lambda v$. Hence $(\mathsf{T} - \lambda I)(v) = 0$, so $\mathsf{T} - \lambda I$ has nonzero kernel and is not invertible.

(k) **False.** For example,
$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$
has characteristic polynomial $t^2 + 1$, so it has no real eigenvalues.

(l) **True.** Since $AA^{-1} = I$, taking determinants gives $\det(A)\det(A^{-1}) = \det(I) = 1$. Hence $\det(A^{-1}) = (\det A)^{-1}$.

(o) **True.** If A is upper triangular, then $\chi_A(t)$ is the product of the terms $t - a_{ii}$, so the eigenvalues are the diagonal entries.

(p) **True.** This is the Cayley-Hamilton theorem.

(q) **False.** For example, if $A = I$, then $A - I = 0$, so $p(t) = t - 1$ satisfies $p(A) = 0$. But if A is 2×2, then $\chi_A(t) = (t-1)^2$, and $t - 1$ is not a multiple of $\chi_A(t)$.

(r) **False.** In general,
$$\dim(\mathsf{U} + \mathsf{W}) = \dim(\mathsf{U}) + \dim(\mathsf{W}) - \dim(\mathsf{U} \cap \mathsf{W}).$$
For example, if $\mathsf{U} = \mathsf{W}$, then $\dim(\mathsf{U} + \mathsf{W}) = \dim(\mathsf{U})$, not $2\dim(\mathsf{U})$.

(s) **True.** If $v \in \operatorname{im}(\mathsf{T})$, then $v = \mathsf{T}(w)$ for some w. Thus $\mathsf{T}(v) = \mathsf{T}^2(w) = 0$, so $v \in \ker(\mathsf{T})$.

(t) **True.** Any two bases of a finite-dimensional vector space have the same number of vectors. This number is $\dim \mathsf{V}$.

Question 2. Let $T \colon V \to V$ be a diagonalizable linear transformation.

(a) Define $\det T$.

(b) Let λ be an eigenvalue of T and $E_\lambda \subset V$ its associated eigenspace. Show that E_λ is T-invariant.

(c) Show that

$$\det(T) = \prod_\lambda \det(T|_{E_\lambda}). \tag{1}$$

Solution. (a) Choose a basis β of V, and let $[T]_\beta$ be the matrix of T with respect to this basis. Then define $\det(T) = \det([T]_\beta)$. This definition does not depend on the choice of basis, because matrices for the same linear transformation in different bases are similar, and similar matrices have the same determinant.

(b) Let $v \in E_\lambda$. Then $T(v) = \lambda v$. To show that E_λ is T-invariant, we need to show that $T(v) \in E_\lambda$. Since $T(v) = \lambda v$ and E_λ is a subspace, we have $\lambda v \in E_\lambda$. Thus $T(v) \in E_\lambda$, so E_λ is T-invariant.

(c) Since T is diagonalizable, we have a direct sum decomposition

$$V = \bigoplus_\lambda E_\lambda.$$

Choose a basis for each eigenspace E_λ. Putting these bases together gives a basis of V. With respect to this basis, the matrix of T is block diagonal, with one block for each restriction $T|_{E_\lambda}$. Therefore the determinant of T is the product of the determinants of these blocks. Hence

$$\det(T) = \prod_\lambda \det(T|_{E_\lambda}).$$

Equivalently, since $T|_{E_\lambda} = \lambda \mathbb{1}_{E_\lambda}$, we have $\det(T|_{E_\lambda}) = \lambda^{\dim E_\lambda}$, and so $\det(T)$ is the product of the eigenvalues with multiplicity.

Question 3. Let V be an n-dimensional real vector space and let $\omega \in \mathrm{Alt}_n(V, \mathbb{R})$ be an n-linear alternating function. Let $T, S\colon V \to V$ be two linear transformations and assume that $v_1, \dots, v_n$ are linearly independent. Show that:

$$\omega(TSv_1, \dots, TSv_n) = \frac{\omega(Tv_1, \dots, Tv_n)\omega(Sv_1, \dots, Sv_n)}{\omega(v_1, \dots, v_n)}. \tag{2}$$

Solution. Since $v_1, \dots, v_n$ are linearly independent and $\dim V = n$, they form a basis of V. Since ω is alternating and $v_1, \dots, v_n$ are linearly independent, we have $\omega(v_1, \dots, v_n) \neq 0$ unless $\omega = 0$. If $\omega = 0$, then both sides are 0, so the identity is true. Thus we may assume $\omega(v_1, \dots, v_n) \neq 0$.

For any linear map $R\colon V \to V$, the function

$$(u_1, \dots, u_n) \mapsto \omega(Ru_1, \dots, Ru_n)$$

is also n-linear and alternating. Since the space of alternating n-linear functions on an n-dimensional vector space is one-dimensional, there is a scalar λ_R (it only depends on R) such that

$$\omega(Ru_1, \dots, Ru_n) = \lambda_R \omega(u_1, \dots, u_n)$$

for all $u_1, \dots, u_n \in V$.

Applying this to the basis $v_1, \dots, v_n$, we get

$$\lambda_R = \frac{\omega(Rv_1, \dots, Rv_n)}{\omega(v_1, \dots, v_n)}.$$

Now apply this with $R = TS$. We have

$$\omega(TSv_1, \dots, TSv_n) = \lambda_{TS}\omega(v_1, \dots, v_n).$$

Also, $\lambda_{TS} = \lambda_T \lambda_S$ since applying the formula above twice gives

$$\omega(TSu_1, \dots, TSu_n) = \lambda_T \omega(Su_1, \dots, Su_n) = \lambda_T \lambda_S \omega(u_1, \dots, u_n).$$

Therefore

$$\omega(TSv_1, \dots, TSv_n) = \lambda_T \lambda_S \omega(v_1, \dots, v_n).$$

The final result follows.

Question 4. Let $\mathsf{V} = \mathsf{P}_2$ and define $\mathsf{T}\colon \mathsf{V} \to \mathsf{V}$ by the formula

$$\mathsf{T}(f)(x) = f(x+1) - f(1).$$

(a) Compute the matrix of T with respect to the basis $\{1, x, x^2\}$.

(b) Find the characteristic polynomial of T.

(c) Determine the eigenvalues and eigenspaces of T.

(d) Is T diagonalizable?

Solution. Let $f(x) = a+bx+cx^2$. Then $f(x+1) = a+b(x+1)+c(x+1)^2$, while $f(1) = a+b+c$. Therefore

$$\begin{aligned} \mathsf{T}(f)(x) &= a + b(x+1) + c(x+1)^2 - (a+b+c) \\ &= bx + c(x^2 + 2x + 1) - c \\ &= (b+2c)x + cx^2. \end{aligned}$$

(a) We compute $\mathsf{T}(1) = 0$, $\mathsf{T}(x) = x$, and $\mathsf{T}(x^2) = 2x + x^2$. Therefore the matrix of T with respect to the basis $\{1, x, x^2\}$ is

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) Since this matrix is upper triangular, the characteristic polynomial is

$$\chi_{\mathsf{T}}(t) = t(t-1)^2.$$

(c) The eigenvalues are 0 and 1.

For $\lambda = 0$, we solve $\mathsf{T}(f) = 0$. Since $\mathsf{T}(a + bx + cx^2) = (b+2c)x + cx^2$, we get $c = 0$ and $b = 0$. Thus $f = a$, so

$$\mathsf{E}_0 = \operatorname{span}\{1\}.$$

For $\lambda = 1$, we solve $\mathsf{T}(f) = f$. Thus $(b+2c)x+cx^2 = a+bx+cx^2$. Comparing coefficients gives $a = 0$ and $2c = 0$, so $c = 0$. Thus $f = bx$, so

$$\mathsf{E}_1 = \operatorname{span}\{x\}.$$

(d) No. The eigenspaces have total dimension $\dim \mathsf{E}_0 + \dim \mathsf{E}_1 = 1 + 1 = 2$, but $\dim \mathsf{V} = 3$. Therefore T does not have a basis of eigenvectors and is not diagonalizable.

Question 5. (a) State the definition of an *isomorphism* (denoted $\cong$) between two vector spaces.

(b) Let Vect be the set of all finite dimensional vector spaces. Show that the notion of isomorphism $\cong$ is an *equivalence relation* on this set.

(c) Produce a bijection of sets

$$\delta\colon \mathrm{Vect}/\!\cong \longrightarrow \mathbb{Z}_{\geq 0} \tag{3}$$

where $\mathbb{Z}_{\geq 0}$ is the set of non-negative integers.

Solution. (a) An *isomorphism* between vector spaces V and W is a linear map $T\colon V \longrightarrow W$ which is bijective. Equivalently, T is an isomorphism if there exists a linear map $S\colon W \longrightarrow V$ such that $ST = \mathbb{1}_V$ and $TS = \mathbb{1}_W$. If such a map exists, we write $V \cong W$.

(b) We show that $\cong$ is reflexive, symmetric, and transitive.

First, $\cong$ is reflexive because $\mathbb{1}_V\colon V \longrightarrow V$ is an isomorphism. Therefore $V \cong V$.

Second, $\cong$ is symmetric. If $V \cong W$, then there is an isomorphism $T\colon V \longrightarrow W$. Since T is bijective and linear, its inverse $T^{-1}\colon W \longrightarrow V$ is also linear. Therefore $W \cong V$.

Third, $\cong$ is transitive. If $U \cong V$ and $V \cong W$, then there are isomorphisms $T\colon U \longrightarrow V$ and $S\colon V \longrightarrow W$. The composition $ST\colon U \longrightarrow W$ is linear and bijective, so it is an isomorphism. Therefore $U \cong W$.

Hence $\cong$ is an equivalence relation.

(c) Define

$$\delta\colon \mathrm{Vect}/\!\cong \longrightarrow \mathbb{Z}_{\geq 0}$$

by

$$\delta([V]) = \dim V.$$

This is well-defined because if $V \cong W$, then $\dim V = \dim W$.

The map δ is injective because if $\delta([V]) = \delta([W])$, then $\dim V = \dim W$. Any two finite-dimensional real vector spaces of the same dimension are isomorphic, so $[V] = [W]$.

The map δ is surjective because for every $n \in \mathbb{Z}_{\geq 0}$, the vector space $\mathbb{R}^n$ has dimension n. Thus $\delta([\mathbb{R}^n]) = n$.

Therefore δ is a bijection.

Question 6. Consider

$$A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \mu & 1 \\ 0 & 0 & \nu \end{bmatrix}.$$

For which λ, μ, ν is the matrix A diagonalizable?

Solution. Since A is upper triangular, its eigenvalues are λ, μ, ν. We consider cases depending on which of these numbers are equal.

If λ, μ, ν are all distinct, then A has three distinct eigenvalues. Therefore A is diagonalizable. This is the first case.

Case two is to suppose exactly two of them are equal. This splits up into three subcases:

- If $\lambda = \mu \neq \nu$, then the eigenspace for λ is $\ker(A - \lambda I)$. We have

  $$A - \lambda I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & \nu - \lambda \end{bmatrix}.$$

  Solving $(A - \lambda I)x = 0$ gives $x_2 = 0$ and $x_3 = 0$, so the eigenspace for λ has dimension 1. But the algebraic multiplicity of λ is 2, so A is not diagonalizable.

- On the other hand, if $\mu = \nu \neq \lambda$, then

  $$A - \mu I = \begin{bmatrix} \lambda - \mu & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

  Solving $(A - \mu I)x = 0$ gives $x_3 = 0$ and $(\lambda - \mu)x_1 + x_2 = 0$. Thus the eigenspace for μ has dimension 1. But the algebraic multiplicity of μ is 2, so A is not diagonalizable.

- If $\lambda = \nu \neq \mu$, then the repeated eigenvalue is λ and

  $$A - \lambda I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & \mu - \lambda & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

  Solving $(A - \lambda I)x = 0$ gives $x_2 = 0$ and $(\mu - \lambda)x_2 + x_3 = 0$, so $x_3 = 0$. Thus the eigenspace for λ is $\operatorname{span}\{e_1\}$, which has dimension 1. Since the algebraic multiplicity of λ is 2, A is not diagonalizable.

The last case is $\lambda = \mu = \nu$. Then

$$A - \lambda I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The eigenspace has dimension 1, so A is not diagonalizable.

Therefore A is diagonalizable exactly when λ, μ, ν are all distinct.