

# MA 442 - practice problems for exam 1

February 22

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A collection of problems designed to help you prepare for midterm exam 1. These problems are by no means exhaustive, but they are in the same style that will appear on the exam.

**Question 1.** Let  $V$  be a finite-dimensional vector space over  $\mathbb{R}$ . In class we (will) show that the set of *all* linear maps  $V \rightarrow \mathbb{R}$  is naturally a vector space; we denote this vector space by  $\text{Hom}(V, \mathbb{R})$ . Express

$$\dim \text{Hom}(V, \mathbb{R})$$

in terms of  $\dim V$ . (Hint: Given a basis for  $V$  can you construct a basis for  $\text{Hom}(V, \mathbb{R})$ ?)

*Solution.* We will show that  $\dim \text{Hom}(V, \mathbb{R}) = \dim V$ . Let  $\dim V = n$ , and fix a basis  $\{v_i\}$  for  $V$ . For each  $i = 1, \dots, n$  we will define a linear transformation  $\varphi_i: V \rightarrow \mathbb{R}$ . Note that since we have fixed a basis, to define  $\varphi_i$  on all of  $V$  it suffices to define it only on the basis elements. So, define

$$\varphi_i(v_j) = \begin{cases} 1, & i = j \\ 0, & \text{else} \end{cases} \quad (1)$$

We will show that  $\{\varphi_1, \dots, \varphi_n\}$  form a basis for  $\text{Hom}(V, \mathbb{R})$ . First, we show that this set generates. Suppose  $\varphi: V \rightarrow \mathbb{R}$  is an arbitrary linear transformation. Since it is a linear transformation, it is completely determined by its behavior on a basis. Thus,  $\varphi$  is completely determined by its values on the vectors  $\{v_1, \dots, v_n\}$ . Call these values  $a_i \stackrel{\text{def}}{=} \varphi(v_i)$  for  $i = 1, \dots, n$ . Observe that

$$\varphi = \sum_i a_i \varphi_i. \quad (2)$$

Indeed, if we evaluate the left hand side at  $v_j$  then this is  $a_j$  by definition. On the other hand, since  $\varphi_i(v_j)$  is 1 when  $i = j$  and 0 otherwise, the right hand side agrees. Thus, we have shown that the collection  $\{\varphi_i\}$  generates  $\text{Hom}(V, \mathbb{R})$ .

It remains to show that this collection of vectors is linearly independent. Suppose we have a linear relation

$$\lambda_1 \varphi_1 + \dots + \lambda_n \varphi_n = \mathbf{0} \quad (3)$$

where the right hand side is the zero linear transformation. If we evaluate this at  $v_1$  then the left hand side is  $\lambda_1$ . Thus  $\lambda_1 = 0$ . Similarly, we can evaluate at  $v_2, v_3, \dots, v_n$  to see that  $\lambda_2 = \lambda_3 = \dots = \lambda_n = 0$  as well. This shows that  $\{\varphi_i\}$  is indeed a basis.

**Question 2.** Let  $V$  be any vector space and suppose that  $B = \{v_1, v_2, v_3, v_4\}$  is an ordered basis. Define the linear transformation  $T: V \rightarrow V$  by the formulas

$$T(v_1) = v_1 + v_2, \quad T(v_2) = v_2 + v_3, \quad T(v_3) = v_3 + v_4, \quad T(v_4) = v_4.$$

Find the matrix representation of  $T$  with respect to this ordered basis. (So, compute  $[T]_B$ .)

*Solution.*

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

**Question 3.** Suppose that  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  linear transformation with the property that  $\dim \ker T = 1$ . Prove that there exists two bases  $\beta$  and  $\gamma$  of  $\mathbb{R}^2$  such that the matrix representation of  $T$  is

$$[T]_{\beta}^{\gamma} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \quad (4)$$

*Solution.* By assumption, there exists a nonzero  $v \neq 0$  such that  $T(v) = 0$ . In particular  $\{v\}$  is a linearly independent subset of  $\mathbb{R}^2$ . We can extend to a basis  $\{v, u\}$  of  $\mathbb{R}^2$ , which we will reorder to view as the ordered basis  $\beta = \{u, v\}$ . Notice that  $T(u) \neq 0$  since  $\dim \ker T = 1$  and  $u, v$  are linearly independent by construction. Thus  $\{T(u)\}$  is a linearly independent subset of  $\mathbb{R}^2$ . We can, in turn, extend it to an ordered basis  $\gamma = \{T(u), w\}$  of  $\mathbb{R}^2$ . These choices of bases result in the desired matrix.

**Question 4.** Recall that  $\mathbb{R}[x]$  is the vector space of all real polynomials. Let  $E \subset \mathbb{R}[x]$  be the subset of *even* polynomials (so, only even powers of  $x$  can appear, like  $3 + 4x^6 + 9x^{38}$ ). Let  $U \subset \mathbb{R}[x]$  be the subset of *odd* polynomials (so, only odd powers of  $x$  can appear like  $2x + x^7 + 6x^{29}$ .)

- Prove that both  $E$  and  $U$  are subspaces of  $\mathbb{R}[x]$ .
- Prove that  $\mathbb{R}[x] = E \oplus U$ .

*Solution.* (b) We need to show two things: (1)  $\mathbb{R}[x] = E + U$  and (2)  $E \cap U = \{0\}$ . To see (1) note that for any polynomial we can write

$$a_0 + a_1x + a_2x^2 + \cdots = (a_0 + a_2x^2 + a_4x^4 + \cdots) + (a_1x + a_3x^3 + a_5x^5 + \cdots). \quad (5)$$

This shows (1). Now, consider an arbitrary polynomial like on the left hand side. If it is odd then  $a_0 = a_2 = a_4 = \cdots = 0$ . If it is even then  $a_1 = a_3 = \cdots = 0$ . Thus a polynomial is even and odd if and only if it is the zero polynomial. This proves (2).

**Question 5.** Prove that there exists a linear transformation  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$  such that  $T(1, 1) = (1, 0, 2)$ ,  $T(2, 3) = (1, 1, 3)$ . Also, compute  $T(7, 10)$ .

*Solution.* Observe that  $\{(1, 1), (2, 3)\}$  is a basis for  $\mathbb{R}^2$ . So, we know there exists a unique linear transformation with  $T(1, 1) = (1, 0, 2)$ ,  $T(2, 3) = (1, 1, 3)$ . Now

$$(7, 10) = (1, 1) + 3 \cdot (2, 3), \quad (6)$$

thus

$$T(7, 10) = T(1, 1) + 3T(2, 3) = (1, 0, 2) + 3 \cdot (1, 1, 3) = \boxed{(4, 3, 11)}. \quad (7)$$

**Question 6.** This question has two parts.

- Suppose that  $W_1$  and  $W_2$  are two two-dimensional subspaces in  $\mathbb{R}^3$ . Prove that their intersection must contain a nonzero vector. In other words, prove that  $W_1 \cap W_2 \neq \{0\}$ .

- (b) Provide an example of two two-dimensional subspaces in  $\mathbb{R}^4$  which intersect at single point.

In other words, two planes cannot intersect at a single point in  $\mathbb{R}^3$  but they can in  $\mathbb{R}^4$ !

*Solution.* (a) We have proven that

$$\dim W_1 + \dim W_2 - \dim W_1 \cap W_2 = \dim(W_1 + W_2). \quad (8)$$

Filling this in with  $\dim W_1 = \dim W_2 = 2$  this becomes (after rearranging)

$$\dim W_1 \cap W_2 = 4 - \dim(W_1 + W_2). \quad (9)$$

But, since  $W_1 + W_2 \subset \mathbb{R}^3$  we know that  $\dim(W_1 + W_2) \leq 3$ . Thus  $\dim W_1 \cap W_2 \geq 1$  follows from the equality above.

- (b) Consider the subspaces

$$W_1 \stackrel{\text{def}}{=} \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\} \quad (10)$$

and

$$W_2 \stackrel{\text{def}}{=} \text{span} \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \quad (11)$$

**Question 7.** Let  $\mathbb{R}[x]$  denote the vector space of real-valued polynomials. Consider the linear maps

$$\partial, \partial^2, \partial^3: \mathbb{R}[x] \rightarrow \mathbb{R}[x] \quad (12)$$

defined by the formulas  $\partial(f) = f'$ ,  $\partial^2(f) = f''$ ,  $\partial^3(f) = f'''$ . Show that the four-element subset

$$\{1, \partial, \partial^2, \partial^3\} \subset \text{Hom}(\mathbb{R}[x], \mathbb{R}[x])$$

is linearly independent.

*Solution.* Suppose we have a linear relation

$$\lambda_0 \mathbb{1} + \lambda_1 \partial + \lambda_2 \partial^2 + \lambda_3 \partial^3 = \mathbf{0}. \quad (13)$$

Remember, this is a linear relation in the vector space  $\text{Hom}(\mathbb{R}[x], \mathbb{R}[x])$ , in particular everything in the above line is itself a linear transformation. Let's evaluate this expression on the polynomial  $1 \in \mathbb{R}[x]$ . Since this is a constant function, all its derivatives are zero, so the left hand side is simply  $\lambda_0$  and the right hand side is zero; so  $\lambda_0 = 0$ . Similarly, let's evaluate this on the polynomial  $x$ . Since  $\partial(x) = 1$  and  $\partial^2(x) = \partial^3(x) = 0$ , this implies that  $\lambda_1 = 0$ . Next, plug in  $x^2$ . The left hand side is  $2\lambda_2$ ; so  $\lambda_2 = 0$ . Finally, plug in  $x^3$ . The left hand side is  $6\lambda_3$ ; so  $\lambda_3 = 0$ . This shows linear independence.

**Question 8.** Suppose that  $V$  is a real vector space (a vector space defined over the scalars  $\mathbb{R}$ ). Also, suppose that  $W \subset V$  is a subspace with a finite number of elements. Show that  $W = \{0\}$ .

**Question 9.** Let  $P_3$  be the vector space of real-valued polynomials which are at most cubic. Let  $W \subset P_3$  be the subset of polynomials  $f$  such that  $f(2) = 0$ .

(a) Show that  $W$  is a subspace.

(b) Find a basis for  $W$ .

*Solution.* (a) Define the function

$$T: P_3 \rightarrow \mathbb{R} \quad (14)$$

by the formula  $T(f) = f(2)$ . Note that  $T$  is linear:  $T(f + \lambda g) = (f + \lambda g)(2) = f(2) + \lambda g(2)$ . Thus,  $\ker T \subset P_3$  is a subspace. But, note that  $\ker T = W$  the subspace defined in the problem. (b) By the dimension theorem we know that

$$\dim \ker T = \dim P_3 - \dim \operatorname{Im} T = 4 - 1 = 3. \quad (15)$$

where we have used the fact that  $T$  is surjective (prove this). So, we know that a basis must consist of three elements in  $P_3$ . We show that such a basis is  $\{x - 2, (x - 2)^2, (x - 2)^3\}$  by showing that this subset is linearly independent in  $P_3$ . Indeed, suppose

$$\lambda_1(x - 2) + \lambda_2(x - 2)^2 + \lambda_3(x - 2)^3 = 0. \quad (16)$$

Plugging in  $x = 1$  we get the relation

$$-\lambda_1 + \lambda_2 - \lambda_3 = 0 \quad (17)$$

Or,  $\lambda_1 = \lambda_2 - \lambda_3$ . Plugging in  $x = 3$  we get the relation

$$\lambda_1 + \lambda_2 + \lambda_3 = 0. \quad (18)$$

Or,  $\lambda_1 = -\lambda_2 - \lambda_3$ . So, we have shown that

$$\lambda_1 = \lambda_2 - \lambda_3 = -\lambda_2 - \lambda_3. \quad (19)$$

This implies that  $\lambda_2 = 0$ . Thus  $\lambda_1 = -\lambda_3$ . Finally, to our original equation we plug in  $x = 0$  to get

$$-2\lambda_1 - 8\lambda_3 = 0. \quad (20)$$

Since  $\lambda_1 = -\lambda_3$  this shows that  $\lambda_1 = \lambda_3 = 0$  as well.

**Question 10.** Can you find a linear transformation  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  such that  $\operatorname{Im} T = \ker T$ ? If yes, construct an explicit example. If no, prove that there cannot exist one.

*Solution.* Define  $T(a, b) = (b, 0)$ . Then  $(b, 0) = (0, 0)$  implies that  $b = 0$ . This means that  $\ker T = \operatorname{span}\{(1, 0)\}$  (the  $x$ -axis). On the other hand,  $(1, 0)$  also spans the image, so  $\operatorname{Im} T = \operatorname{span}\{(1, 0)\}$  as well.

**Question 11.** The collection of vectors

$$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}.$$

is a basis for  $\mathbb{R}^2$ . Express the standard unit vectors  $\{e_1, e_2\}$  in terms of these basis vectors.

*Solution.* Call the two vectors  $\{u, v\}$ . For  $e_1$  we need to solve

$$\begin{aligned}1 &= a + 3b \\ 0 &= 2a + 4b.\end{aligned}$$

The second equation says that  $a = -2b$ . The first equation then becomes  $1 = -2b + 3b = b$  so  $b = 1$ . Thus  $e_1 = -2u + v$ . For  $e_2$  we need to solve

$$\begin{aligned}0 &= a + 3b \\ 1 &= 2a + 4b.\end{aligned}$$

First equation now says  $a = -3b$  and second equation is  $1 = -6b + 4b = -2b$  or  $b = -1/2$ . Thus  $e_2 = (3/2)u - (1/2)v$ .