MA 442 - practice problems for exam 2

March 29

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A collection of problems designed to help you prepare for midterm exam 2. These problems are by no means exhaustive, but they are in the same style that will appear on the exam.

Question 1. (a) Describe all 2×2 matrices J such that $J^2 = -\mathbb{1}_{2\times 2}$.

(b) Find a 2×2 real matrix A, other than the identity matrix, such that $A^5 = \mathbb{1}$.

Question 2. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be an arbitrary 2×2 matrix. Define e^A to be the 2×2 matrix

$$e^A = \mathbb{1}_{2\times 2} + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \cdots = \sum_{n\geq 0} \frac{1}{n!}A^n. \tag{1}$$

(a) Consider the function $M \colon \mathbb{R} \to \mathrm{M}_{2\times 2}$ defined by

$$M(t) = e^{tA}. \tag{2}$$

Prove that $M'(t) = AM(t)$ and that $M(0) = \mathbb{1}_{2\times 2}$.

(b) Let $f \colon \mathbb{R} \to \mathbb{R}$ be the function

$$f(t) = \det M(t). \tag{3}$$

Prove that $f'(t) = (a + d)f(t)$.

(c) Conclude that $\det(e^A) = e^{\mathrm{tr}(A)}$.

Question 3. Suppose that A is a square matrix and D is a square matrix (possibly different sizes). Show that

$$\det \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} = \det(A)\det(D). \tag{4}$$

Question 4. A *nilpotent* matrix is a square matrix A such that $A^N = 0$ for some integer N (possibly very large). Show that if A is nilpotent then $\det(\mathbb{1} + A) = 1$. (Hint: first show that A is non invertible.)

Question 5. Suppose that A is a 2×2 matrix such that $A^2 = \mathbb{1}_{2\times 2}$. Show that

$$e^A = \alpha \mathbb{1}_{2\times 2} + \beta A \tag{5}$$

for some scalars $\alpha, \beta \in \mathbb{R}$. Find the values of these scalars. (For definition of matrix exponential see equation (1) in question 2.)

Question 6. Each of the following parts are separate problems and are unrelated to one another.

(a) If λ is an eigenvalue of a matrix A, then show that λ^2 is an eigenvalue for A^2.

(b) More generally, show that if p is any polynomial then $p(\lambda)$ is an eigenvalue for $p(A)$.

(c) Show that if A is nilpotent then $\lambda = 0$ is the only eigenvalue of A.

(d) Suppose that B is invertible and λ is an eigenvalue of B. Show that λ^{-1} is an eigenvalue for B^{-1}.

Question 7. For each of the following matrices or transformations: (i) find the eigenvalues and eigenvectors, and (ii) Determine whether the matrix is diagonalizable.

(a) $\begin{bmatrix} 4 & 1 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

(b) $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$.

(c) $\mathsf{T}\colon \mathsf{P}_2 \longrightarrow \mathsf{P}_2$, $\mathsf{T}(f)(x) = xf'(x) - f(x)$.

(d) $\mathsf{T}\colon \mathsf{P}_2 \longrightarrow \mathsf{P}_2$, $\mathsf{T}(f)(x) = xf'(x) + x^2 f''(x)$.

Question 8. Suppose $\mathsf{T}, \mathsf{S}\colon \mathsf{V} \longrightarrow \mathsf{V}$ are two linear transformations between the same vector space. Show that

$$\operatorname{rank}(\mathsf{T} \circ \mathsf{S} - \mathsf{S} \circ \mathsf{T}) \leq 2 \min\{\operatorname{rank}(\mathsf{T}), \operatorname{rank}(\mathsf{S})\}. \tag{6}$$

(Hint: First show that $\operatorname{rank}(\mathsf{T}_1 + \mathsf{T}_2) \leq \operatorname{rank}(\mathsf{T}_1) + \operatorname{rank}(\mathsf{T}_2)$ for any $\mathsf{T}_1, \mathsf{T}_2$.)

Question 9. Consider the following 3×3 matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & c \\ 0 & 0 & \mu. \end{bmatrix} \tag{7}$$

where λ, μ, c are scalars.

(a) Suppose $\lambda = \mu \neq 1$. For which a, if any, is the matrix A diagonalizable?

(b) Suppose $\mu = 1$ and $\lambda \neq 1$. For which c, if any, is the matrix A diagonalizable?