Vectors in $\mathbb{R}^2$, $\mathbb{R}^3$ (and $\mathbb{R}^n$).

- Euclidean space $\mathbb{R}^n$ is the set of n-tuples of real numbers

$$\{(a_1, \ldots, a_n) \mid a_i \in \mathbb{R}\}$$

- On the other hand consider the set V of all "arrows" which begin at $\underline{0} = (0, \ldots, 0) \in \mathbb{R}^n$ and end at some other point. We call V the set of <u>vectors</u>.
- There is an *bijection* of sets

$$\mathbb{R}^n \xrightarrow{\cong} V$$

which sends a point $P = (a_1, \ldots, a_n)$ to the vector which ends at P. In other words

$$P \longmapsto \overrightarrow{OP}$$

[We will review the concept of sets, maps, bijection, etc... in discussion section. Also, see Appendix A].

Write $\vec{OP} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in V$ where

$$P = (a_1, \dots, a_n) \in \mathbb{R}^n .$$

Recall some familiar operations :

- Vector addition

$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix}$$

- Scalar Multiplication

  If $\lambda \in \mathbb{R}$ then

$$\lambda \cdot \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \lambda a_1 \\ \vdots \\ \lambda a_n \end{bmatrix} .$$

We will now abstract these operations (and their intrinsic properties) into the notion of a

$\star$ **Vector space**. $\star$

• Comment on * fields *. A field $\mathbb{F}$ is an algebraic object for which we will do linear algebra "over". We will spend some more time w/ fields in this course, but for now we usually take $\mathbb{F} = \mathbb{R}$ the real numbers.

{ See Appendix C for a background on fields }

A field $\mathbb{F}$ forms the "scalars". Other examples of fields include $\mathbb{F} = \mathbb{C}$, the complex numbers, or $\mathbb{F} = \mathbb{F}_2 = \{0, 1\}$ (the field w/ two elements).

We move on to our main definition.

Dfn: A vector space over a field $\mathbb{F}$ is a set

$$V = \{\text{the set of "vectors"}\}$$

together with operations

- Addition: if $v, w \in V$ then $v + w \in V$.

- Scalar multiplication: if $v \in V$, $\lambda \in \mathbb{F}$ then $\lambda \cdot v \in V$.

These operations must satisfy the following EIGHT axioms:

VS 1) For all $v, w \in V$ have $v + w = w + v$.

VS 2) For all $v, w, u \in V$ have

$$(v+w)+u = v + (w+u).$$

VS 3) There is $0 \in V$ s.t. $v + 0 = v$ for all $v \in V$. (The zero vector)

VS 4) For all $v \in V$ there exists $\tilde{v} \in V$ s.t. $v + \tilde{v} = 0$.

VS 5) For all $v \in V$, $1 \cdot v = v$, where $1 \in \mathbb{F}$ is the unit.

VS 6) For all $\lambda, \mu \in \mathbb{F}$ and $v \in V$ have

$$(\lambda \mu) \cdot v = \lambda \cdot (\mu \cdot v).$$

VS 7) For all $\lambda \in \mathbb{F}$, and $v, w \in V$ have

$$\lambda \cdot (v + w) = \lambda \cdot v + \lambda \cdot w$$

VS 8) For all $\lambda, \mu \in \mathbb{F}$ and $v \in V$ have

$$(\lambda + \mu) \cdot v = \lambda \cdot v + \mu \cdot v.$$

Ex: For any integer n we can look at

$$\mathbb{F}^n = \{ (a_1, \dots, a_n) \mid a_i \in \mathbb{F}, i = 1, \dots, n \}.$$

In the familiar case $\mathbb{F} = \mathbb{R}$ this is $\mathbb{R}^n$. Write elements of $\mathbb{F}^n$ in "column vector" notation

$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{F}^n$$

Then the familiar operations

$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix}$$

$$\lambda \cdot \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \lambda a_1 \\ \vdots \\ \lambda a_n \end{bmatrix}, \quad \lambda \in \mathbb{F}.$$

Endow $\mathbb{F}^n$ w/ the structure of a vector space over $\mathbb{F}$.

Q: What is the zero vector?

$\underline{Ex}$: A sequence in $\mathbb{F}$ is an ordered, countably infinite collection of elements of $\mathbb{F}$:

$$\underline{a} = (a_1, a_2, a_3, \dots)$$

where each $a_i \in \mathbb{F}$. Denote the set of all such sequences by $\mathbb{F}^\infty$. Then $\mathbb{F}^\infty$ has the natural structure of a vector space over $\mathbb{F}$.

[Heuristically : " $\lim_{n\to\infty} \mathbb{F}^n = \mathbb{F}^\infty$ "].

$\underline{Ex}$: A polynomial in $\mathbb{F}$ is an expression of the form

$$a_0 + a_1 x + a_2 x^2 + \cdots$$

where $a_i \in \mathbb{F}$ and all but finitely many of the a_i's are zero. [So only a finite power of x can appear.]

Then, we can define the "addition"

$$(a_0 + a_1 x + a_2 x^2 + \cdots) + (b_0 + b_1 x + b_2 x^2 + \cdots)$$

$$= (a_0 + b_0) + (a_1 + b_1) x + (a_2 + b_2) x^2 + \cdots$$

And "scalar multiplication"

$$\lambda \cdot (a_0 + a_1 x + a_2 x^2 + \cdots)$$

$$= \lambda a_0 + \lambda a_1 x + \lambda a_2 x^2 + \cdots .$$

These operations satisfy VS1 – VS8. So the set of all polynomials form a vector space. We denote it

$$\mathbb{F}[x]$$

[Can you extend this to polynomials in more than one variable :

$$\mathbb{F}[x, y, z, \ldots] \quad ? \quad].$$

- Some observations. V : vector space over $\mathbb{F}$.

Lem: If $v + z = v' + z$ for $v, v', z \in V$ then $v = v'$.

Pf: We know by VS4 that there exists $w \in V$ such that $z + w = 0$.

Thus

$$\begin{aligned}
v &= v + 0 && (\text{VS } 3) \\
&= v + (z + w) \\
&= (v + z) + w && (\text{VS } 2) \\
&= (v' + z) + w \\
&= v' + (z + w) && (\text{VS } 2) \\
&= v' + 0 \\
&= v' && (\text{VS } 3).
\end{aligned}$$ ∎

Notice that we have used axioms VS2, VS3, VS4 to prove this.

Cor: The zero vector $0 \in V$ is the unique vector s.t. $v + 0 = v$ for all $v \in V$.

Pf: If $v + 0 = v + w = v$ then $w = 0$ by the lemma. $\boxtimes$

Cor: If $v \in V$ then $-v = -1 \cdot v \in V$ is the unique vector s.t.

$$v + (-v) = 0.$$