Let V be a vector space.

Thm: a) $0 \cdot v = 0$. The zero vector.

b) $(-\lambda) \cdot v = -(\lambda \cdot v) = \lambda(-v)$
for all $\lambda \in \mathbb{F}$, $v \in V$.

c) $\lambda \cdot 0 = 0$ for all $\lambda \in \mathbb{F}$

zero vector.

Pf: a) By VS 8, VS 3, VS 1:

$$0 \cdot v + 0 \cdot v = (0+0) \cdot v = 0 \cdot v = 0 \cdot v + 0$$

$\Rightarrow$ $0 \cdot v = 0$ by the cancellation lemma.

b) From last time, we know $-(\lambda \cdot v)$ is the unique vector for which

$$\lambda \cdot v + (-(\lambda \cdot v)) = 0.$$

So, if $\lambda \cdot v + (-\lambda) \cdot v = 0$ we have that $(-\lambda) \cdot v = -(\lambda \cdot v)$. In particular,

$$(-1) \cdot v = -v$$

c) Exercise. $\blacksquare$

Exercises for §1.2:

1, 4(e) - 4(h), 7, 8, 10, 12, 13, 17, 20, 21.

• <u>Subspaces.</u>

A subspace of a vector space is a <u>subset</u> which is, itself, a vector space. Here is the precise definition.

V is a vector space

<u>Dfn</u>: A subset $W \subset V$ is a <u>subspace</u> if it is a vector space according to the operations of addition and multiplication in V.

$\boxed{*}$ Note: if $W \subset V$ is a subspace then it <u>must</u> contain the zero vector!

<u>Ex</u>: $W = \{0\}$ = the single element set containing the zero vector

$\cap$

V

is a subspace.

<u>Ex</u>: $W = V$ is a subspace.

$\cap$

V

- To check whether $W \subset V$ is a subspace:

1) $v + w \in W$ whenever $v, w \in W$
(closed under addition)

2) $\lambda v \in W$ whenever $v \in W$ and $\lambda \in \mathbb{F}$.
(closed under scalar mult.)

3) $W \ni 0$.

4) Each vector $v \in W$ has $-v \in W$ as well.

In fact, 4) is repetitive.

<u>Theorem</u>: $W \subset V$ is a subspace if and only if the following three conditions are satisfied:

a) $0 \in W$

b) $v + w \in W$ whenever $v, w \in W$

c) $\lambda v \in W$ whenever $v \in W$, $\lambda \in \mathbb{F}$.

Pf: ($\Rightarrow$) Assume W is a subspace. Then (b) and (c) automatically hold. Also, we know there is $0' \in W$ s.t. $v + 0' = v$ for all $v \in W$. But also $v + 0 = v$ for all $v \in V$. So, if $v \in W$ then

$$v + 0' = v + 0 \Rightarrow 0 = 0'.$$

Thus $0 = 0' \in W$.

($\Leftarrow$) Conversely, suppose (a)-(c) hold. To see W is subspace we need to show that the additive inverse of each $v \in W$ is in W. But by (c) we have $-v = (-1) \cdot v \in W$. Hence W is a subspace of V. $\blacksquare$

Ex: $V = \mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$.

We know $\mathbb{R}^3$ is a vector space using the usual rules.

$\rightsquigarrow$ $W = \{z = 0\} = \{(x, y, 0) \mid x, y \in \mathbb{R}\}$

"xy-plane" $\subset \mathbb{R}^3$ is a subspace.

$\underline{Ex}$: $V = \mathbb{R}[x]$ = set of all polynomials with real coefficients.

We know this is a vector space.

$W_d = \left\{ \begin{array}{c} \text{set of all polynomials} \\ \text{of degree } \leq d \end{array} \right\}$

$\underline{E.g}$: $W_2 = \{ a_0 + a_1 x + a_2 x^2 \,/\, a_0, a_1, a_2 \in \mathbb{R} \}$

Then, $W_d \subset \mathbb{R}[x]$ is a subspace.

- Intersection: If A is a set and

$$B, C \subset A$$

are subsets. The $\underline{\text{intersection}}$ is

$B \cap C = \left\{ \begin{array}{c} \text{elements } a \in A \text{ s.t.} \\ a \in B \ \underline{\text{and}} \ a \in C \end{array} \right\}.$

Theorem: If V is a vector space and $W, Z \subset V$ are two subspaces. Then

$$W \cap Z \subset V$$

is a subspace.

Pf: Let $v, w \in W \cap Z$. Then

$$v + w \in W \text{ and } v + w \in Z$$

$$\Rightarrow v + w \in W \cap Z.$$

Similarly, $\lambda v \in W \cap Z$ for all $\lambda \in \mathbb{F}$. $\blacksquare$