Linear combinations

__Dfn__: Let V be a vector space over $\mathbb{F}$. A vector $v \in V$ is said to be a linear combination of vectors $\{u_1, \dots, u_n\}$ if it can be written

$$v = \underbrace{a_1 u_1 + \cdots + a_n u_n}_{n\text{-terms}} .$$

(or

$$v = \sum_{i=1}^{n} a_i u_i . \quad)$$

where $a_1, \dots, a_n \in \mathbb{F}$.

__Ex__: The vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$:

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} .$$

The problem of solving linear systems of equations is the problem of finding if and how a vector v is a linear combination of other vectors $u_1, \dots, u_n$.

$\underline{Ex}$: Take $V = \mathbb{R}^3$. Is $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \in \mathbb{R}^3$ a linear combination of the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$?

In other words, can we find $a_1, a_2 \in \mathbb{R}$ such that

$$a_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$$\Leftrightarrow \boxed{\begin{aligned} a_1 + a_2 &= 2 \\ a_1 + 2a_2 &= 1 \\ a_1 - a_2 &= 3 \end{aligned}}$$

This is a <u>system</u> of <u>linear</u> <u>equations</u>

Can we solve this example? To do this, lets add the third equation to the first:

$$2a_1 = 5 \Rightarrow a_1 = 5/2 .$$

But then eqn 1 says $5/2 + a_2 = 2$ or $a_2 = -1/2$. So eqn 1, 2 force $a_1 = 5/2,\ a_2 = -1/2$.

Does this solve eqn 2?

$$\begin{array}{c} a_1 + 2a_2 \overset{?}{=} 1 \\ \| \\ 5/2 - 1 \qquad 3/2 \neq 1 . \\ \| \\ 3/2 \end{array}$$

Thus, $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ is <u>not</u> a linear combination of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$.

Span: Let V be a vector space. Let

$$S \subset V$$

be a subset of vectors. The span of S is the set

$$\operatorname{Span}(S) = \left\{ \begin{array}{c} \text{all linear combinations} \\ \text{of vectors in } S \end{array} \right\}.$$

Theorem: For any $S \subset V$ the span

$$\operatorname{span}(S) \subset V$$

is a subspace. Moreover, $\operatorname{span}(S)$ is the smallest subspace which contains S.

Pf: If $S = \emptyset$ then

$$\operatorname{span}(\emptyset) = \{0\}, \quad \text{which is subspace.}$$

If $S \neq \emptyset$ then S contains a vector $z \in S$.
So $0 \cdot z = \vec{0}$ is in Span S. If $x, y \in \operatorname{Span}(S)$ then

$$x = a_1 u_1 + \cdots + a_m u_m$$

$$y = b_1 v_1 + \cdots + b_n v_n$$

for some $a_i, b_j \in \mathbb{F}$, $u_i, v_j \in S$.

Then

$$x + y = a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots + b_n v_n$$

is still a linear combination of vectors in S. Thus $x+y \in \mathrm{Span}(S)$. Similarly $\lambda \cdot x \in \mathrm{Span}(S)$ for any $\lambda \in \mathbb{F}$. Thus $\mathrm{Span}(S)$ is subspace.

Now, suppose $W \subset V$ is a subspace which contains S. If $w \in \mathrm{Span}(S)$ then

$$w = a_1 w_1 + \cdots + a_k w_k$$

for some $a_1, \ldots, a_k \in \mathbb{F}$ and $w_1, \ldots, w_k \in S$. Since $S \subset W$ we see $w_1, \ldots, w_k \in W$ as well. But, since W is subspace we see that $w \in W$. Thus $\mathrm{Span}(S) \subset W$. Since W was an arbitrary subspace which contains S, we see $\mathrm{Span}(S)$ is the <u>smallest</u> subspace which contains S.

$\blacksquare$

Dfn: A subset $S \subset V$ spans V if

$$\operatorname{span}(S) = V.$$

Ex: $V = \mathbb{R}^3$. Then, the three element subset:

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

spans $\mathbb{R}^3$.

Pf: For any $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{R}^3$ we need to

find a_1, a_2, a_3 s.t.

$$a_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + a_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

You can directly check that

$$\begin{cases} a_1 = \frac{1}{2}(v_1 + v_2 - v_3) \\ a_2 = \frac{1}{2}(v_1 - v_2 + v_3) \\ a_3 = \frac{1}{2}(-v_1 + v_2 + v_3) \end{cases}$$

solves this eqn. So S spans $\mathbb{R}^3$. ▨

Q: Typically, one can find many subsets which span a given subspace. The goal in the next lecture is deciding how to find such a subset which is as "small as possible".

- Sum of subspaces.

Given two subspaces $W_1, W_2 \subset V$ define

$W_1 + W_2$ to be the smallest subspace which contains W_1 and W_2.

Prop: Let $S_1, S_2 \subset V$ be arbitrary subsets. Then

$$\operatorname{Span}(S_1 \cup S_2) = \operatorname{Span}(S_1) + \operatorname{Span}(S_2).$$

Pf: To show that two subsets are equal, we will show that they both contain the other.

Suppose first that $v \in \text{Span}(S_1 \cup S_2)$. Thus:

$$v = \underbrace{\sum_i \lambda_i x_i}_{x} + \underbrace{\sum_j \mu_j y_j}_{y}$$

where $x_i \in S_1$, $y_j \in S_2$ and $\lambda_i, \mu_j \in \mathbb{R}$.

But, clearly $x \in \text{Span}(S_1)$ and $y \in \text{Span}(S_2)$.

Thus

$$v = x + y \in \text{Span}(S_1) + \text{Span}(S_2)$$

This shows $\text{Span}(S_1 \cup S_2) \subset \text{Span}(S_1) + \text{Span}(S_2)$.

A very similar argument shows the reverse inclusion.

Try it as an exercise

$\blacksquare$

· Linear dependence.

Dfn: A subset S of a vector space V is called linearly dependent if there exists a finite number of distinct vectors $u_1, \dots, u_m$ in S and scalars $\lambda_1, \dots, \lambda_m \in \mathbb{F}$, not all zero, such that:

$$\lambda_1 u_1 + \dots + \lambda_m u_m = 0.$$

(We sometimes say that "the vectors in S are linearly dependent.)

· A set of vectors S is called linearly independent if it is not linearly dependent.

Ex: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix} \right\}$ is a set of linearly dependent vectors in $\mathbb{R}^3$.

We will show this on the next page.

Indeed, we need to show that there exists scalars $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ such that

$$\lambda_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \lambda_3 \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix} = 0 .$$

$$\Longleftrightarrow \quad \begin{aligned} \lambda_1 + 0 + 2\lambda_3 &= 0 \\ \lambda_1 + \lambda_2 - \lambda_3 &= 0 \\ 0 + \lambda_2 - 3\lambda_3 &= 0 . \end{aligned}$$

One solution is $\lambda_1 = 2$, $\lambda_2 = -3$, $\lambda_3 = -1$.

(How did I solve this? Is this the on solution to these eqns ?).

Ex : A set $S = \{u\}$ consisting of a single vector is linearly independent $\Longleftrightarrow$ $u \neq \vec{0} \in V$.

Note that a set $S = \{u_1, \ldots, u_m\}$ is linearly independent $\Longleftrightarrow$ the only solution to

$$\lambda_1 u_1 + \cdots + \lambda_m u_m = 0$$

is $\lambda_1 = \lambda_2 = \cdots = \lambda_m = 0$.

Ex: Recall the vector space $V = \mathbb{R}[x]$ of real-valued polynomials. Let

$$p_k(x) = x^k + x^{k+1} + \cdots + x^n$$

for $k = 0, 1, \ldots, n$. Then the set

$$\{p_0, p_1, \ldots, p_n\}$$ is linearly independent.

(exercise).

Theorem: Let V be a vector space and suppose

$S_1 \subset S_2 \subset V$, are subsets.

If S_1 is linearly dependent then so is S_2.

Pf: Exercise. $\blacksquare$

Cor: Let V be a vector space, and let $S_1 \subset S_2$.
If S_2 is linearly independent then so is S_1.

Pf: This is simply the negation of the theorem. $\blacksquare$

Theorem: Let $S \subset V$ be a linearly independent subset of vectors and suppose

$$v \in V \setminus S \quad ,$$

is a vector not in S. Then the subset $S \cup \{v\}$ is linear dependent $\iff v \in \operatorname{Span} S$.

Pf: If $S \cup \{v\}$ is linearly dependent we can find $u_1, \dots, u_m \in S$ and scalars $\lambda, \lambda_1, \dots, \lambda_m \in \mathbb{F}$ such that

$$\lambda v + \lambda_1 u_1 + \cdots + \lambda_m u_m = \vec{0} .$$

Claim: $\lambda \neq 0$. For if $\lambda = 0$ this would say $\{u_1, \dots u_m\} \subset S$ are linearly dependent which is contradiction.

Thus $\lambda \neq 0$, so

$$v = -\lambda^{-1} (\lambda_1 u_1 + \cdots + \lambda_m u_m) .$$

Thus $v \in \operatorname{Span} S$.

Conversely, assume $v \in \operatorname{Span} S$. Then $\exists$ $\lambda_1, \dots \lambda_m \in \mathbb{F}$ and some vectors $u_1, \dots u_m \in S$ such that

$$v = \lambda_1 u_1 + \cdots + \lambda_m u_m.$$

(Note one of $\lambda_1, \dots \lambda_m$ must be nonzero. Why?)

But then

$$(-1)v + \lambda_1 u_1 + \cdots + \lambda_m u_m = \vec{0}$$

is a linear dependence. $\blacksquare$