Bases and dimension

We have met <u>two</u> different concepts involving a set of vectors.

$$S = \{u_1, \dots, u_n\}$$

1) $\text{Span}(S) = \left\{ \begin{array}{l} \text{subspace generated by} \\ \text{linear combinations of } S \end{array} \right\}$

2) S is <u>linearly</u> <u>independent</u> if

$$\lambda_1 u_1 + \cdots + \lambda_n u_n = 0$$

$$\Leftrightarrow \lambda_1 = \lambda_2 = \cdots = \lambda_n = 0.$$

<u>Dfn</u>: A subset $B \subset V$ is a basis if:

1) $\text{Span}(B) = V$ (so, B "generates" V)

2) B is linearly independent.

Recall that since $\mathcal{B}$ generates S that any $v \in V$ can be expressed as:

$$v = \lambda_1 u_1 + \cdots + \lambda_n u_n \qquad (*)$$

for some $\lambda_i \in \mathbb{F}$, and some $u_i \in \mathcal{B}$

Thm: If $\mathcal{B}$ is basis then the way to express v in $(*)$ is unique.

Pf: Suppose we could alternatively write

$$v = \lambda_1' u_1 + \cdots + \lambda_n' u_n \quad , \quad \lambda_i' \in \mathbb{F}$$

[Why can I write down the same vectors u_i?]

But then: $v - v = \vec{0} = (\lambda_1 - \lambda_1') u_1 + \cdots + (\lambda_n - \lambda_n') u_n$

by lin. independence

$$\Rightarrow \quad \lambda_1 - \lambda_1' = \lambda_2 - \lambda_2' = \cdots = \lambda_n - \lambda_n' = 0$$

$$\Rightarrow \quad \lambda_1 = \lambda_1', \ \lambda_2 = \lambda_2', \ldots, \lambda_n = \lambda_n'. \qquad \blacksquare$$

- A basis may not be finite.

<u>Ex</u> : $V = \mathbb{R}[x]$ the set of all real polynomials in a single variable.

The set

$$B = \{1, x, x^2, \dots\}$$

is an infinite basis.

<u>Theorem</u> : If a vector space V is generated by a finite set S then there exists a finite basis for V.

<u>Pf</u> : If $S = \phi$ or $\{\vec{0}\}$ then $V = \{\vec{0}\}$, and we are done.

Otherwise S contains $u_1 \in S$ w/ $u_1 \neq \vec{0}$.

Keep picking vectors in S (if possible) so that $B = \{u_1, \dots, u_k\} \subset S$ are linearly independent, but adjoining any other vector in S results in a lin dependent subset.

<u>Claim</u> : B is a basis.

{ We must only show that B generates V.

It suffices to show $S \subset \text{span}(B)$ since then $\text{span}(S) = V = \text{span}(B)$. So, let $v \in S$. If $v \in B \subset S$ then certainly $v \in \text{span}(B)$. So, suppose $v \notin B$. Then, by assumption, $B \cup \{v\}$ is linearly dependent. So by the theorem from last time we know $v \in \text{span}(B)$. Thus $S \subset \text{span}(B)$.]

$\blacksquare$

<u>Ex</u>: $S = \left\{ \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}, \begin{bmatrix} 8 \\ -12 \\ 20 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 7 \\ 2 \\ 6 \end{bmatrix} \right\}$.

Theorem [Replacement] Let V be a vector space. And let $G \subset V$ be a subset which generates V, and $\# G = n$. Let $L \subset V$ be a linearly independent subset, and $\# L = m$.

Then $m \leq n$ and there exists $H \subset G$ w/ $\# H = n - m$ such that $L \cup H$ generates V.

Proof later.

Cor: If V has a finite basis, then every other basis for V has the same size.

Pf: Sps that $\mathcal{B}$ and $\mathcal{B}'$ are bases and $\# \mathcal{B} = n$. If $\# \mathcal{B}' > n$ then there exists $S \subset \mathcal{B}'$ w/ $\# S = n+1$. Since S is linearly independent, the replacement theorem implies $n+1 \leq n$, a contradiction. So, $\# \mathcal{B}' = m < \infty$ and $m \leq n$. Now, reverse roles of $\mathcal{B}$ and $\mathcal{B}'$ to see that $n \leq m$ as well. So $m = n$. ∎

A vector space V is called <u>finite dimensional</u> if the number of vectors in a basis B is finite.

The <u>dimension of V</u> is

$$\boxed{\dim V = \# B}$$

where B is any basis.

<u>Ex</u>: $\dim \mathbb{R}^n = n$.

<u>Ex</u>: Recall that

$$P_n \subset \mathbb{R}[x]$$

is the subset of all polynomials of degree $\leq n$. Then

$$\dim P_n = n+1.$$

A basis for P_n is:

$$B = \{1, x, x^2, \dots, x^n\}.$$

<u>Prop</u>: Let V be a v.s. w/ $\dim V = n$.

a) Any finite generating set for V contains at least n vectors. A generating set which contains exactly n vectors is a basis.

b) Any linearly independent subset of V that contains exactly n vectors is a basis.

c) Any linearly independent subset of V can be extended to a basis by adding vectors to it.

<u>Pf</u>: (a) Sps $G \subset V$ generates V and is finite. By the theorem some subset $H \subset G$ is a basis for V. The result follows.

(b) Sps $L \subset V$ is lin. independent, $\# L = n$. Let $\mathcal{B}$ be any basis for V. By the replacement theorem, there is $H \subset \mathcal{B}$ containing $n - n = 0$ vectors s.t. $L \cup H$ generates V. So $H = \phi$, $L \cup H = L$ generates V.

(c) If $L \subset V$ is lin. ind., $\# L = m \leq n$, the replacement theorem $\Rightarrow$ there is $H \subset B$ containing $n-m$ vectors s.t. $L \cup H$ generates V. Now, $L \cup H$ contains at most $m + n - m = n$ vectors, so (a) $\Rightarrow$ $L \cup H$ contains exactly n vectors and is a basis. $\blacksquare$

Ex: $V = P_2 \subset \mathbb{R}[x]$ is the vector space of polynomials which are at most quadratic.

Consider the subset

$$G = \{ x^2 + 3x - 2,\ 2x^2 + 5x - 3,\ -x^2 - 4x + 4 \}.$$

For any $a, b, c \in \mathbb{R}$ we have:

$$\begin{aligned} ax^2 + bx + c = & (-8a + 5b + 3c)(x^2 + 3x - 2) \\ & + (4a - 2b - c)(2x^2 + 5x - 3) \\ & + (-a + b + c)(-x^2 - 4x + 4) \end{aligned}$$

So, G generates P_2.

On the other hand, by (a) we see that G is a basis for P_2 since $\dim P_2 = 3$.