Optional projects

Last update: March 3

The following is a list of projects which you may turn in to replace some number of quiz scores. The exact number depends on the project. Please note that each problem has it's own deadline, I will only accept solutions before this deadline. Solutions must be written in complete sentences and all details must be provided.

Project 1: *Lagrange polynomials.* Replaces one quiz score, deadline March 18.

Let $c_0, c_1, \dots, c_n$ be distinct real numbers. Define, for $i = 0, \dots, n$ the functions $f : \mathbb{R} \to \mathbb{R}$ by the formula

$$f_i(x) = \prod_{k \neq i} \frac{x - c_k}{c_i - c_k}. \tag{1}$$

(a) Show for each i that $f_i(x) \in \mathsf{P}_n$.

(b) Show that $\{f_0, \dots, f_n\} \subset \mathsf{P}_n$ is linearly independent and hence a basis.

(c) Suppose that $g \in \mathsf{P}_n$ is any polynomial such that $g(c_0) = g(c_1) = \cdots = g(c_n) = 0$. Show that g is the zero polynomial.

Project 2: *The dimension of* **Hom(V, W)**. Replaces one quiz score, deadline March 18

In class, we have seen that the set $\mathrm{Hom}(\mathsf{V}, \mathsf{W})$ of all linear transformations from V to W form a vector space with the obvious rules of addition and scalar multiplication. In this problem, you will compute $\dim \mathrm{Hom}(\mathsf{V}, \mathsf{W})$.

(a) Set $\dim \mathsf{V} = n, \dim \mathsf{W} = m$. Let $\beta = \{v_i\}$ be a basis for V and $\{w_j\}$ be a basis for W. For any $i = 1, \dots, n$ and $j = 1, \dots, m$ define $\varphi_{i,j} : \beta \to \mathsf{W}$ by the formula:

$$\varphi_{i,j}(v_k) = \begin{cases} w_j, & i = k \\ 0, & \text{else} \end{cases} \tag{2}$$

Argue why $\varphi_{i,j}$ determines a linear transformation $\mathsf{V} \to \mathsf{W}$.

(b) Show that the collection $\{\varphi\}_{i=1,\dots,n; j=1,\dots,m} \subset \mathrm{Hom}(\mathsf{V}, \mathsf{W})$ is linearly independent.

(c) Show that the collection from part (b) is a basis. Conclude that

$$\dim \mathrm{Hom}(\mathsf{V}, \mathsf{W}) = (\dim \mathsf{V}) \cdot (\dim \mathsf{W}). \tag{3}$$

Project 3: *Multilinear functions.* Replaces one quiz score, deadline March 25.

If A, B are any sets, we can form a new set called the product set denoted $A \times B$. It's defined by

$$A \times B = \{(a, b) \mid a \in A, b \in B\}. \tag{4}$$

(a) If $\#A = n$ and $\#B = m$ then $\#(A \times B) = ?$.

(b) Now, consider three sets A, B and C. And let $f\colon A \times B \to C$ be any function. If $a \in A$ is a fixed element, define $f_{a,}\colon B \to C$ by the formula $f_{a,}(b) = f(a,b)$. And, if $b \in B$ is a fixed element, define $f_{,b}\colon A \to C$ by the formula $f_{,b}(a) = f(a,b)$. Show that $f\colon A \times B \to C$ is surjective if there exists some $a \in A$ such that $f_{a,}\colon B \to C$ is surjective.

(c) Consider three vector spaces V, W, and U. Define a *bilinear* transformation to be a function

$$\mathsf{T}\colon \mathsf{V} \times \mathsf{W} \to \mathsf{U} \tag{5}$$

such that the two conditions hold:

(i) $\mathsf{T}_{v,}\colon \mathsf{W} \to \mathsf{U}$ is a linear transformation for any vector $v \in \mathsf{V}$.

(ii) $\mathsf{T}_{,w}\colon \mathsf{V} \to \mathsf{U}$ is a linear transformation for any vector $w \in \mathsf{W}$.

Show that the function

$$\mathsf{T}\colon \mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R} \tag{6}$$

defined by

$$\mathsf{T}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right) = x_1y_2 - x_2y_1 + 3x_1y_3$$

is bilinear.

(d) Generalize the definition above to define the notion of a *k-multilinear transformation* $\mathsf{T}\colon \mathsf{V}_1 \times \cdots \times \mathsf{V}_k \to \mathsf{U}$. Provide one nontrivial example in the case $k = 3$.

Project 4: *Determinant in dimension two.* Replaces two quiz scores, deadline March 25

This problem uses the concept of a bilinear map defined in the previous project. Let V be a two-dimensional vector space. A *two-form* on V is a bilinear map

$$\omega\colon \mathsf{V} \times \mathsf{V} \to \mathbb{R} \tag{7}$$

with the property that $\omega(v,w) = -\omega(w,v)$ for all $v, w \in \mathsf{V}$.

(a) Show that there exists a nontrivial two-form on V (Hint: pick a basis).

(b) Let ω_0 be the nontrivial two-form you found in part (a). Show that any other two-form ω satisfies $\omega(v,w) = \delta\omega_0(v,w)$ for all $v, w \in \mathsf{V}$ where δ is some scalar (the scalar is independent of v, w).

(c) Suppose that $\mathsf{T}\colon \mathsf{V} \to \mathsf{V}$ is a linear transformation and let ω be any two-form on V. Define the function

$$\omega_{\mathsf{T}}\colon \mathsf{V} \times \mathsf{V} \to \mathbb{R} \tag{8}$$

by the formula $\omega_{\mathsf{T}}(v,w) = \omega(\mathsf{T}(v), \mathsf{T}(w))$. Show that ω_T is a two-form.

(d) By part (c), there exists a δ_{T} such that

$$\omega_{\mathsf{T}}(v,w) = \delta_{\mathsf{T}}\omega(v,w). \tag{9}$$

Show that δ_{T} is well-defined; that is, it does not depend on the two-form ω and so only depends on T.

(e) Consider the example of $V = \mathbb{R}^2$ and the linear transformation

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix} \tag{10}$$

for some scalars a, b, c, d. Compute δ_T in this example. Does the expression look familiar?

Project 5: *Fields of dreams.* Replaces one quiz score, deadline March 25.

This project requires you to read the definition of a **field** given in appendix C of the textbook. So, it is worth more.

(a) Let $V_{\mathbb{C}}$ be a vector space over the field $\mathbb{C}$ of complex numbers

$$\mathbb{C} \stackrel{\text{def}}{=} \{a + bi \mid a, b \in \mathbb{R}\}. \tag{11}$$

Define the function $J : V_{\mathbb{C}} \to V_{\mathbb{C}}$ by the formula $J(x) = ix$. Explain why this function J is a $\mathbb{C}$-linear transformation.

(b) The field of real numbers $\mathbb{R}$ is a subset (in fact, subfield) of the field of complex numbers $\mathbb{C}$. This means, that any vector space $V_{\mathbb{C}}$ over $\mathbb{C}$ can be viewed as a vector space $V_{\mathbb{R}}$ over $\mathbb{R}$. (The sets $V_{\mathbb{C}}$ and $V_{\mathbb{R}}$ are the same, we are just viewing them as vector spaces over different fields.) Let $\{v_1, \ldots, v_n\}$ be a basis for $V_{\mathbb{C}}$ as a complex vector space. Show that $\{v_1, J(v_1), v_2, J(v_2), \cdots, v_n, J(v_n)\}$ is a basis for $V_{\mathbb{R}}$ as a real vector space. Conclude the formula $\dim_{\mathbb{R}} V_{\mathbb{R}} = 2 \dim_{\mathbb{C}} V_{\mathbb{C}}$.

(c) These next two questions pertain to the finite field $\mathbb{F}_2 = \{0, 1\}$ with two elements (see example 4 from appendix C). Let V be a vector space over the field $\mathbb{F}_2$. Show that if V is finite-dimensional then it is finite. Compute the size of V in terms of its dimension.

(d) Consider the vector space $\mathbb{F}_2^3$ over $\mathbb{F}_2$. As a set

$$\mathbb{F}_2^3 = \{(a, b, c) \mid a, b, c \in \mathbb{F}_2\}. \tag{12}$$

Count the total number of subspaces of $\mathbb{F}_2^3$.

Project 6: *Counting points on a subspace.* Replaces four quizzes scores, no deadline. (Requires in-person presentation.)

Let $H \subset \mathbb{R}^n$ denote the set of points in $\mathbb{R}^n$ whose coordinates are either 0 or 1. For example when $n = 3$ the points $(0, 1, 1), (1, 1, 1), (0, 1, 0), (0, 0, 0)$ are all in H yet $(2, 0, 1)$ is *not* in H. Show that if $W \subset \mathbb{R}^n$ is a k-dimensional subspace then

$$\#(H \cap W) \leq 2^k. \tag{13}$$

After that, produce a subspace W which contains the maximum (so 2^k) points in H.