It is easily seen that T is diagonalizable with eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 3$. Furthermore, the corresponding eigenspaces coincide with the subspaces W_1, W_2, and W_3 of Example 9. Thus Theorem 5.11 provides us with another proof that $\mathsf{R}^4 = \mathsf{W}_1 \oplus \mathsf{W}_2 \oplus \mathsf{W}_3$. ♦

EXERCISES

1. Label the following statements as true or false.

(a) Any linear operator on an n-dimensional vector space that has fewer than n distinct eigenvalues is not diagonalizable.

(b) Two distinct eigenvectors corresponding to the same eigenvalue are always linearly dependent.

(c) If λ is an eigenvalue of a linear operator T, then each vector in E_λ is an eigenvector of T.

(d) If λ_1 and λ_2 are distinct eigenvalues of a linear operator T, then $\mathsf{E}_{\lambda_1} \cap \mathsf{E}_{\lambda_2} = \{0\}$.

(e) Let $A \in \mathsf{M}_{n\times n}(F)$ and $\beta = \{v_1, v_2, \ldots, v_n\}$ be an ordered basis for F^n consisting of eigenvectors of A. If Q is the $n \times n$ matrix whose jth column is v_j $(1 \leq j \leq n)$, then $Q^{-1}AQ$ is a diagonal matrix.

(f) A linear operator T on a finite-dimensional vector space is diagonalizable if and only if the multiplicity of each eigenvalue λ equals the dimension of E_λ.

(g) Every diagonalizable linear operator on a nonzero vector space has at least one eigenvalue.

The following two items relate to the optional subsection on direct sums.

(h) If a vector space is the direct sum of subspaces $\mathsf{W}_1, \mathsf{W}_2, \ldots, \mathsf{W}_k$, then $\mathsf{W}_i \cap \mathsf{W}_j = \{0\}$ for $i \neq j$.

(i) If

$$\mathsf{V} = \sum_{i=1}^{k} \mathsf{W}_i \quad \text{and} \quad \mathsf{W}_i \cap \mathsf{W}_j = \{0\} \quad \text{for } i \neq j,$$

then $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2 \oplus \cdots \oplus \mathsf{W}_k$.

2. For each of the following matrices $A \in \mathsf{M}_{n\times n}(R)$, test A for diagonalizability, and if A is diagonalizable, find an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

(a) $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ **(b)** $\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$ **(c)** $\begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$

(d) $\begin{pmatrix} 7 & -4 & 0 \\ 8 & -5 & 0 \\ 6 & -6 & 3 \end{pmatrix}$ **(e)** $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$ **(f)** $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$

(g) $\begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}$

3. For each of the following linear operators T on a vector space V, test T for diagonalizability, and if T is diagonalizable, find a basis β for V such that $[\mathsf{T}]_\beta$ is a diagonal matrix.

(a) $\mathsf{V} = \mathsf{P}_3(R)$ and T is defined by $\mathsf{T}(f(x)) = f'(x) + f''(x)$, respectively.

(b) $\mathsf{V} = \mathsf{P}_2(R)$ and T is defined by $\mathsf{T}(ax^2 + bx + c) = cx^2 + bx + a$.

(c) $\mathsf{V} = \mathsf{R}^3$ and T is defined by

$$\mathsf{T}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_2 \\ -a_1 \\ 2a_3 \end{pmatrix}.$$

(d) $\mathsf{V} = \mathsf{P}_2(R)$ and T is defined by $\mathsf{T}(f(x)) = f(0) + f(1)(x + x^2)$.

(e) $\mathsf{V} = \mathsf{C}^2$ and T is defined by $\mathsf{T}(z, w) = (z + iw, iz + w)$.

(f) $\mathsf{V} = \mathsf{M}_{2\times 2}(R)$ and T is defined by $\mathsf{T}(A) = A^t$.

4. Prove the matrix version of the corollary to Theorem 5.5: If $A \in \mathsf{M}_{n\times n}(F)$ has n distinct eigenvalues, then A is diagonalizable.

5. State and prove the matrix version of Theorem 5.6.

6. **(a)** Justify the test for diagonalizability and the method for diagonalization stated in this section.

(b) Formulate the results in (a) for matrices.

7. For

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \in \mathsf{M}_{2\times 2}(R),$$

find an expression for A^n, where n is an arbitrary positive integer.

8. Suppose that $A \in \mathsf{M}_{n\times n}(F)$ has two distinct eigenvalues, λ_1 and λ_2, and that $\dim(\mathsf{E}_{\lambda_1}) = n - 1$. Prove that A is diagonalizable.

9. Let T be a linear operator on a finite-dimensional vector space V, and suppose there exists an ordered basis β for V such that $[\mathsf{T}]_\beta$ is an upper triangular matrix.

(a) Prove that the characteristic polynomial for T splits.

(b) State and prove an analogous result for matrices.

The converse of (a) is treated in Exercise 32 of Section 5.4.

10. Let T be a linear operator on a finite-dimensional vector space V with the distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ and corresponding multiplicities $m_1, m_2, \ldots, m_k$. Suppose that β is a basis for V such that $[\mathsf{T}]_\beta$ is an upper triangular matrix. Prove that the diagonal entries of $[\mathsf{T}]_\beta$ are $\lambda_1, \lambda_2, \ldots, \lambda_k$ and that each λ_i occurs m_i times $(1 \le i \le k)$.

11. Let A be an $n \times n$ matrix that is similar to an upper triangular matrix and has the distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ with corresponding multiplicities $m_1, m_2, \ldots, m_k$. Prove the following statements.

(a) $\operatorname{tr}(A) = \sum_{i=1}^{k} m_i \lambda_i$

(b) $\det(A) = (\lambda_1)^{m_1}(\lambda_2)^{m_2} \cdots (\lambda_k)^{m_k}$.

12. Let T be an invertible linear operator on a finite-dimensional vector space V.

(a) Recall that for any eigenvalue λ of T, λ^{-1} is an eigenvalue of T^{-1} (Exercise 8 of Section 5.1). Prove that the eigenspace of T corresponding to λ is the same as the eigenspace of T^{-1} corresponding to λ^{-1}.

(b) Prove that if T is diagonalizable, then T^{-1} is diagonalizable.

13. Let $A \in \mathsf{M}_{n\times n}(F)$. Recall from Exercise 14 of Section 5.1 that A and A^t have the same characteristic polynomial and hence share the same eigenvalues with the same multiplicities. For any eigenvalue λ of A and A^t, let E_λ and E'_λ denote the corresponding eigenspaces for A and A^t, respectively.

(a) Show by way of example that for a given common eigenvalue, these two eigenspaces need not be the same.

(b) Prove that for any eigenvalue λ, $\dim(\mathsf{E}_\lambda) = \dim(\mathsf{E}'_\lambda)$.

(c) Prove that if A is diagonalizable, then A^t is also diagonalizable.

14. Find the general solution to each system of differential equations.

(a) $\begin{aligned} x' &= x + y \\ y' &= 3x - y \end{aligned}$ **(b)** $\begin{aligned} x_1' &= 8x_1 + 10x_2 \\ x_2' &= -5x_1 - 7x_2 \end{aligned}$

(c) $\begin{aligned} x_1' &= x_1 + x_3 \\ x_2' &= x_2 + x_3 \\ x_3' &= 2x_3 \end{aligned}$

15. Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

be the coefficient matrix of the system of differential equations

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ x_2' &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ &\vdots \\ x_n' &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n. \end{aligned}$$

Suppose that A is diagonalizable and that the distinct eigenvalues of A are $\lambda_1, \lambda_2, \ldots, \lambda_k$. Prove that a differentiable function $x\colon R \to \mathsf{R}^n$ is a solution to the system if and only if x is of the form

$$x(t) = e^{\lambda_1 t}z_1 + e^{\lambda_2 t}z_2 + \cdots + e^{\lambda_k t}z_k,$$

where $z_i \in \mathsf{E}_{\lambda_i}$ for $i = 1, 2, \ldots, k$. Use this result to prove that the set of solutions to the system is an n-dimensional real vector space.

16. Let $C \in \mathsf{M}_{m\times n}(R)$, and let Y be an $n \times p$ matrix of differentiable functions. Prove $(CY)' = CY'$, where $(Y')_{ij} = Y'_{ij}$ for all i, j.

Exercises 17 through 19 are concerned with *simultaneous diagonalization.*

Definitions. *Two linear operators* T *and* U *on a finite-dimensional vector space* V *are called* ***simultaneously diagonalizable*** *if there exists an ordered basis* β *for* V *such that both* $[\mathsf{T}]_\beta$ *and* $[\mathsf{U}]_\beta$ *are diagonal matrices. Similarly,* $A, B \in \mathsf{M}_{n\times n}(F)$ *are called* ***simultaneously diagonalizable*** *if there exists an invertible matrix* $Q \in \mathsf{M}_{n\times n}(F)$ *such that both* $Q^{-1}AQ$ *and* $Q^{-1}BQ$ *are diagonal matrices.*

17. **(a)** Prove that if T and U are simultaneously diagonalizable linear operators on a finite-dimensional vector space V, then the matrices $[\mathsf{T}]_\beta$ and $[\mathsf{U}]_\beta$ are simultaneously diagonalizable for any ordered basis β.

(b) Prove that if A and B are simultaneously diagonalizable matrices, then L_A and L_B are simultaneously diagonalizable linear operators.

18. **(a)** Prove that if T and U are simultaneously diagonalizable operators, then T and U commute (i.e., $\mathsf{TU} = \mathsf{UT}$).

(b) Show that if A and B are simultaneously diagonalizable matrices, then A and B commute.

The converses of (a) and (b) are established in Exercise 25 of Section 5.4.

19. Let T be a diagonalizable linear operator on a finite-dimensional vector space, and let m be any positive integer. Prove that T and T^m are simultaneously diagonalizable.

Exercises 20 through 23 are concerned with direct sums.

20. Let $\mathsf{W}_1, \mathsf{W}_2, \ldots, \mathsf{W}_k$ be subspaces of a finite-dimensional vector space V such that

$$\sum_{i=1}^{k} \mathsf{W}_i = \mathsf{V}.$$

Prove that V is the direct sum of $\mathsf{W}_1, \mathsf{W}_2, \ldots, \mathsf{W}_k$ if and only if

$$\dim(\mathsf{V}) = \sum_{i=1}^{k} \dim(\mathsf{W}_i).$$

21. Let V be a finite-dimensional vector space with a basis β, and let $\beta_1, \beta_2, \ldots, \beta_k$ be a partition of β (i.e., $\beta_1, \beta_2, \ldots, \beta_k$ are subsets of β such that $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$ and $\beta_i \cap \beta_j = \varnothing$ if $i \neq j$). Prove that $\mathsf{V} = \mathrm{span}(\beta_1) \oplus \mathrm{span}(\beta_2) \oplus \cdots \oplus \mathrm{span}(\beta_k)$.

22. Let T be a linear operator on a finite-dimensional vector space V, and suppose that the distinct eigenvalues of T are $\lambda_1, \lambda_2, \ldots, \lambda_k$. Prove that

$$\mathrm{span}(\{x \in \mathsf{V}\colon x \text{ is an eigenvector of } \mathsf{T}\}) = \mathsf{E}_{\lambda_1} \oplus \mathsf{E}_{\lambda_2} \oplus \cdots \oplus \mathsf{E}_{\lambda_k}.$$

23. Let $\mathsf{W}_1, \mathsf{W}_2, \mathsf{K}_1, \mathsf{K}_2, \ldots, \mathsf{K}_p, \mathsf{M}_1, \mathsf{M}_2, \ldots, \mathsf{M}_q$ be subspaces of a vector space V such that $\mathsf{W}_1 = \mathsf{K}_1 \oplus \mathsf{K}_2 \oplus \cdots \oplus \mathsf{K}_p$ and $\mathsf{W}_2 = \mathsf{M}_1 \oplus \mathsf{M}_2 \oplus \cdots \oplus \mathsf{M}_q$. Prove that if $\mathsf{W}_1 \cap \mathsf{W}_2 = \{0\}$, then

$$\mathsf{W}_1 + \mathsf{W}_2 = \mathsf{W}_1 \oplus \mathsf{W}_2 = \mathsf{K}_1 \oplus \mathsf{K}_2 \oplus \cdots \oplus \mathsf{K}_p \oplus \mathsf{M}_1 \oplus \mathsf{M}_2 \oplus \cdots \oplus \mathsf{M}_q.$$

5.3* MATRIX LIMITS AND MARKOV CHAINS

In this section, we apply what we have learned thus far in Chapter 5 to study the *limit* of a sequence of powers $A, A^2, \ldots, A^n, \ldots$, where A is a square matrix with complex entries. Such sequences and their limits have practical applications in the natural and social sciences.

We assume familiarity with limits of sequences of real numbers. The limit of a sequence of complex numbers $\{z_m\colon m = 1, 2, \ldots\}$ can be defined in terms of the limits of the sequences of the real and imaginary parts: If $z_m = r_m + is_m$, where r_m and s_m are real numbers, and i is the imaginary number such that $i^2 = -1$, then

$$\lim_{m\to\infty} z_m = \lim_{m\to\infty} r_m + i \lim_{m\to\infty} s_m,$$

provided that $\lim\limits_{m\to\infty} r_m$ and $\lim\limits_{m\to\infty} s_m$ exist.