Solutions to selected exercises from §1.2

Problem 12

Let E denote the set of even functions $f : \mathbb{R} \to \mathbb{R}$. We will view this as subset of all functions $\mathbb{R} \to \mathbb{R}$ which we already know has the structure of a vector space using the obvious rules of addition and scalar multiplcation. Recall, we denoted this vector space $\mathcal{F}(\mathbb{R}, \mathbb{R})$. We show that E is a subspace of this vector space (see §1.3 for the definition of subspace). First, we show that E is closed under addition. If f, g are even functions then we have

$$(f + g)(-x) = f(-x) + g(-x) = f(x) + g(x) = (f + g)(x). \tag{1}$$

In the first equality we have used the definition of addition of functions. In the second equality we have used the fact that both f and g are even functions. This shows that $f + g$ is an even function. Similarly, if f is an even function and $\lambda \in \mathbb{R}$ is a scalar, then λf is an even function. Finally, we point out that the zero function is even. So $0 \in \mathsf{E}$. This concludes the proof that E is a subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$. In particular, E is a vector space.

Problem 17

This is *not* a vector space with the indicated definition of scalar multiplication. We refer to VS5 which states that $1 \cdot x = x$ for all $x \in V$. Note that with this definition, however, we have $1 \cdot (a_1, a_2) = (a_1, 0) \neq (a_1, a_2)$. Thus, this is not a vector space structure.