Solutions to selected exercises from §1.3

Problem 19

We prove that these conditions are necessary by proving the negation of the statement. That is, suppose that neither $W_1 \subset W_2$ nor $W_2 \subset W_1$ are true. In other words, there exist a vector $v \in W_1$ for which $v \notin W_2$ as well as vector $w \in W_2$ for which $w \notin W_1$. We will show that $v + w \notin W_1 \cup W_2$. Indeed, suppose by way of contradiction that $v + w$ is in this union. Then either $v+w \in W_1$ or $v+w \in W_2$. If $v+w \in W_1$ then $(v+w)+(-v) = w$ is a linear combination of elements of W_1 therefore is also in W_1, contradiction. Similarly, $v + w$ is not in W_2. We have shown that $W_1 \cup W_2$ is not a subspace.

For the other direction, suppose that $W_1 \subset W_2$. Then $W_1 \cup W_2 = W_2$ which is a subspace. On the other hand, if $W_2 \subset W_1$ then $W_1 \cup W_2 = W_1$ is a subspace.

Problem 23

This problem uses the *sum* of subspaces which was defined in lecture. I give an alternative definition here.

Definition 0.1. Let $W_1, W_2 \subset V$ be two subspaces of a vector space. Define the *sum* $W_1 + W_2$ to be the subspace of V consisting of all linear combinations involving elements of W_1 and W_2.

A useful lemma is the following. You should prove it!

Lemma 0.2. $W_1 + W_2 = \{x + y \mid x \in W_1, y \in W_2\}$.

Now onto the exercise.

(a)

Since $0 \in W_2$ we see that for any $x \in W_1$ that $x + 0 \in W_1 + W_2$. Thus, $W_1 \subset W_1 + W_2$. Similarly, $W_2 \subset W_1 + W_2$.

(b)

Suppose that W is a subspace which contains both W_1 and W_2. If $x \in W_1$ and $y \in W_2$ we see that $x + y \in W$ by the subspace property for W. Thus $W \subset W_1 + W_2$.

Problem 30

We use this definition.

Definition 0.3. We say that $W_1 + W_2$ is the *direct sum* of W_1 and W_2 if $W_1 \cap W_2 = \{0\}$. If this is the case, we write the direct sum as $W_1 \oplus W_2$.

Onto the exercise. For the first direction we prove the negatation. Suppose that $z \in W_1 \cap W_2$ is *not* the zero vector. Then notice that for any $x \in W_1$ and $y \in W_2$, another way to rewrite the element $v = x + y \in W_1 + W_2$ is

$$(x + z) + (y - z) \in W_1 + W_2. \tag{1}$$

Thus, the decomposition of v into a sum of a vector in W_1 and a vector in W_2 is not unique.

Now for the other direction. Suppose that $W_1 \cap W_2 = \{0\}$ and let $v \in W_1 + W_2$. We know that v can be expressed as $v = x + y$ where $x \in W_1$, $y \in W_2$. We need to show that this way to express v is unique. Suppose $x' \in W_1, y' \in W_2$ also satisfy $v = x' + y'$. Then

$$(x - x') + (y - y') = 0 \tag{2}$$

Not that we can rewrite this as $x - x' = y' - y$. The left hand side of the equation is obviously in W_1, so $y - y' \in W_1$ as well. But then $y - y' \in W_1 \cap W_2$. So $y - y' = 0$ or $y = y'$ and hence $x = x'$ as well. We have shown that the decomposition is unique.