Solutions to selected exercises from §1.4

Question 6

Fix $a, b, c\mathbb{R}$. So solve the problem we must find $\lambda_1, \lambda_2, \lambda_3$ such that

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}. \tag{1}$$

In other words $a = \lambda_1 + \lambda_2, b = \lambda_1 + \lambda_3, c = \lambda_2 + \lambda_3$. So

$$a - b + c = 2\lambda_2, a + b - c = 2\lambda_1, -a + b + c = 2\lambda_3. \tag{2}$$

This shows that $\lambda_1, \lambda_2, \lambda_3$ exist and hence the three vectors generated $\mathbb{R}^3$.

Question 11

A vector v span$\{x\}$ is in the span of the singleton set if and only if $v = \lambda x$ for some $\lambda \in \mathbb{F}$. Thus

$$\mathrm{span}\{x\} = \{\lambda x \mid \lambda \in \mathbb{F}\}. \tag{3}$$

This is a line.

Question 12

First, suppose that $\mathrm{W} \subset \mathrm{V}$ is a subspace. We will show that $\mathrm{span}(\mathrm{W}) = \mathrm{W}$. For any set S it is always true that $S \subset \mathrm{span}(S)$, so to prove the assertion we must only show that $\mathrm{span}(\mathrm{W}) \subset \mathrm{W}$. For this, start with $v \in \mathrm{span}(\mathrm{W})$. By definition of span, there exists $\lambda_1, \ldots, \lambda_m \in \mathbb{F}$ and $u_1, \ldots, u_m \in \mathrm{W}$ such that

$$v = \lambda_1 u_1 + \cdots + \lambda_m u_m. \tag{4}$$

On the other hand, since W is a subspace we know that it is closed under arbitrary addition and scalar multiplication. Thus, we see that $v \in \mathrm{W}$ as desired.

In the other direction, assume that $\mathrm{W} \subset \mathrm{V}$ is a subset such that $\mathrm{span}(\mathrm{W}) = \mathrm{W}$. Since we have shown that $\mathrm{span}(S)$ is a subspace for any S, the result follows.

Question 16

In class on 02/03.