Solutions to selected exercises from §1.5

Question 8

This question uses the concept of a field of characteristic 2 which we have not yet covered. Please skip this for now.

Question 9

Suppose that $\{u, v\} \subset \mathrm{V}$ is linearly dependent. That means there exists scalars λ, μ at least one nonzero such that

$$\lambda u + \mu v = \mathbf{0}. \tag{1}$$

Or $\lambda u = -\mu v$. If $\lambda = 0$ then $\mu v = \mathbf{0}$ with $\mu \neq 0$. Thus $v = \mathbf{0}$ and certainly $\mathbf{0}$ is a scalar multiple of any vector. On the other hand, if $\lambda \neq 0$ then $u = -\frac{\mu}{\lambda} v$. Again, one vector is a multiple of the other.

Conversely, suppose that one vector is the multiple of the other, say $v = cu$ for some scalar c. Then $v - cu = \mathbf{0}$ is a linear dependence. Similarly, if $u = cv$, the expression $cv - u = \mathbf{0}$ is linear dependence. In either case, we have proven that $\{u, v\}$ is linearly dependent.

Question 11

This question uses the concept of a field of characteristic 2 which we have not yet covered. Please skip this for now.

Question 14

Suppose that S is linearly dependent and that $S \neq \{\mathbf{0}\}$. Our first case is that $\mathbf{0} \in S$. If this is the case, then we can take $v = \mathbf{0}$ and trivially $v = \mathbf{0}$ is a linear combination of any other vector in S, namely $\mathbf{0} = 0 \cdot u$ for any $u \in S$. The next case is that $\mathbf{0} \notin S$. Since the set S is linearly dependent, there exists vectors $\{u_0, u_1, \ldots, u_n\}$ and scalars $\lambda_0, \ldots, \lambda_n$, not all zero, such that

$$\lambda_0 u_0 + \lambda_1 u_1 + \cdots + \lambda_n u_n = \mathbf{0} \tag{2}$$

We can assume, without loss of generality, that $u_i \neq u_j$ for all $i, j = 1, \ldots, n$. Indeed, suppose for example that $u_{n-1} = u_n$. Then, we can write this expression as

$$\lambda_0 u_0 + \cdots + (\lambda_{n-1} + \lambda_n) u_{n-1} = \mathbf{0} \tag{3}$$

Keep proceeding this way until all vectors u_i are distinct.

Question 20

Suppose that there exists a linear dependence relation. That is, scalars λ, μ such that

$$\lambda e^{rt} + \mu e^{st} = \mathbf{0} \tag{4}$$

where $\mathbf{0}$ is the zero function. In other words, for each $t \in \mathbb{R}$, we have the equation

$$\lambda e^{rt} + \mu e^{rt} = 0. \tag{5}$$

Plugging in $t = 0$ we find that $\lambda + \mu = 0$, or $\lambda = -\mu$. Plugging in $t = 1$ we find that

$$\lambda e^r + \mu e^s = \lambda(e^r - e^s) = 0. \tag{6}$$

Since we started with a linear dependence relation, we know that $\lambda \neq 0$. Thus, for this equation to be true at $t = 1$ we see that $e^r - e^s = 0$, or $e^r = e^s$. But, the exponential is an *injective* function, so this would imply that $r = s$; a contradiction! Thus, the functions e^{rt}, e^{st} are linearly independent.