Solutions to selected exercises from §1.6

Question 4

No, these vectors do not generate P_3 (the set of polynomials, with real coefficients, which are at least cubic). The reason is that $\dim \mathrm{P}_3 = 4$ as we have shown that a basis for this vector space consists of four vectors (for example $\{1, x, x^2, x^3\}$). Since a basis is the minimum sized subset which can generate a vector space, a collection of vectors consisting of less than four vectors can not be generating.

For practice, we will show that these three vectors are linearly independent, however. Suppose $\lambda_1, \lambda_2, \lambda_3$ are scalars such that

$$\mathbf{0} = \lambda_1(x^3 - 2x^2 + 1) + \lambda_2(4x^2 - x + 3) + \lambda_3(3x - 2). \tag{1}$$

This equation can be written as

$$\mathbf{0} = \lambda_1 x^3 + (-2\lambda_1 - 4\lambda_2)x^2 + (-\lambda_2 + 3\lambda_3)x + (\lambda_1 + 3\lambda_2 - 2\lambda_3). \tag{2}$$

For this to be zero, each coefficient must be zero. Thus $\lambda_3 = 0$ comes from vanishing of the cubic term. From the vanishing of the linear term we see that $\lambda_2 = 0$. From the vanishing of the quadratic term we see $\lambda_1 = 0$. Thus, the collection is linear independent.

Question 7

The vectors u_1, u_2 are clearly not parallel, hence $\{u_1, u_2\}$ is linearly independent. Next, we check if adjoining u_3 results in a linearly independent set. Suppose $\lambda_1, \lambda_2, \lambda_3$ are scalars such that

$$\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 = \mathbf{0}.$$

This is equivalent to the system of three equations

$$2\lambda_1 + \lambda_2 - 8\lambda_3 = 0 \tag{3}$$
$$-3\lambda_1 + 4\lambda_2 + 12\lambda_3 = 0 \tag{4}$$
$$\lambda_1 - 2\lambda_2 - 4\lambda_3 = 0. \tag{5}$$

Let's add two times row one to row three to get

$$5\lambda_1 - 20\lambda_3 = 0 \tag{6}$$

or $\lambda_1 = 4\lambda_3$. Plugging back into the original third equation this means $4\lambda_3 - 2\lambda_2 - 4\lambda_3 = 0$ or $\lambda_2 = 0$. This simplifies the second equation to $-3\lambda_1 + 12\lambda_3 = 0$. In other words, the second equation gives us no additional conditions on λ_1, λ_3. Thus, as long as $\lambda_2 = 0$ and $\lambda_3 = \lambda$ and $\lambda_1 = 4\lambda$ for some scalar λ then we have solved these equations. In particular, $(\lambda_1, \lambda_2, \lambda_3) = (4, 0, 1)$ is a nonzero solution, therefore $\{u_1, u_2, u_3\}$ is *linearly dependent.* One can similarly check that $\{u_1, u_2, u_4\}$ is also linearly dependent while $\{u_1, u_2, u_5\}$ is linearly independent.

Question 11

First, we will show that $\{u+v, au\}$ is a basis. Notice, since $\dim \mathsf{V} = 2$ we need to onlyl show that $u+v, au$ are linearly independent. So, suppose that $\lambda_1(u+v)+\lambda_2(au) = \mathbf{0}$ for λ_1, λ_2 scalars. This simplifies to $(\lambda_1 + a\lambda_2)u + \lambda_1 v = 0$. Since $\{u, v\}$ is a basis, hence linearly independent, we see that this implies $\lambda_1 + a\lambda_2 = 0$ and $\lambda_1 = 0$. But, these equations together imply that $\lambda_1 = \lambda_2 = 0$ since $a \neq 0$.

Next, we show that $\{au, bv\}$ is a basis where a, b are nonzero scalars. Again, it suffices to show that this set is linearly independent. Suppose $\lambda_1(au) + \lambda_2(bv) = \mathbf{0}$. This is the same as $(a\lambda_1)u + (b\lambda_2)v = 0$ which implies $a\lambda_1 = b\lambda_2 = 0$. Since a, b are both nonzero this implies $\lambda_1 = \lambda_2 = 0$ hence the set is linearly independent.

Question 20

(a) If $\mathsf{V} = \{0\}$ then the statement is tautological. Suppose $\mathsf{V} \neq \{0\}$. Then, there exists a nonzero vector u_1 in S. So, $\{u_1\}$ is linearly independent. Now, suppose we continue this process until we build a subset $\{u_1, \ldots, u_k\} \subset S$ which is linearly independent and such that adjoining any other vector to $\{u_1, \ldots, u_k\}$ results in a linear dependent subset. We know such a process truncates at finite k since V is finite-dimensional.

We claim that $\{u_1, \ldots, u_k\} \subset S$ is the desired basis. To see this it suffices to show that it generates V. We will first prove the following lemma.

Lemma 0.1. *Suppose $v \in S$. Then $v \in \operatorname{span}\{u_1, \ldots, u_k\}$.*

Proof. If $v \in \{u_1, \ldots, u_k\}$ then the statement is tautological. If $v \in S \setminus \{u_1, \ldots, u_k\}$ then we know that $\{u_1, \ldots, u_k, v\}$ is a linear *dependent* subset, by our construction. Hence, by theorem 1.7 we see that $v \in \operatorname{span}\{u_1, \ldots, u_k\}$. □

We return to part (a). Suppose now that $w \in \mathsf{V}$ is an arbitrary vector. Since S generates, we know that there exists scalars $\lambda_1, \ldots, \lambda_m$ and vectors $v_1, \ldots, v_m \in S$ such that

$$v = \lambda_1 v_1 + \cdots + \lambda_m v_m. \tag{7}$$

By the lemma we know that $v_i \in \operatorname{span}\{u_1, \ldots, u_k\}$. Hence $v \in \operatorname{span}\{u_1, \ldots, u_k\}$ as well. Thus $\{u_1, \ldots, u_k\}$ generates and it is a basis.

(b) If S is finite then we can just apply corollary 2(a) from the book. If S is infinite then it certainly contains at least n vectors. In either case, the statement is proven.

Question 22

Observe that $\mathsf{W}_1 \cap \mathsf{W}_2$ is always a subspace of W_1. Hence $\dim(\mathsf{W}_1 \cap \mathsf{W}_2) \leq \dim \mathsf{W}_1$ by theorem 1.11 in the book. Furthermore, the other part of theorem 1.11 says that this is an equality if and only if $\mathsf{W}_1 \cap \mathsf{W}_2 = \mathsf{W}_1$. We know this can only be true when $\mathsf{W}_1 \subset \mathsf{W}_2$. Thus, the necessary and sufficient condition is that W_1 must be a subspace of W_2.

Question 24

Since f is of degree n it is of the form

$$f = ax^n + h \tag{8}$$

where $a \neq 0$ is a nonzero number and h is a polynomial of degree strictly less than n. We claim that $\{f, f', f'', \ldots, f^{(n)}\}$ is a basis for P_n. The solution of the problem follows immediately.

Note that $f' = nax^{n-1} + h'$ where h' is now a polynomial of degree strictly less than $n-1$. In particular, f' is a polynomial of degree $n-1$. More generally, for $0 \leq k \leq n$ we see that the kth derivative is

$$f^{(k)} = n(n-1)\cdots(n-k+1)x^{n-k} + h^{(k)} \tag{9}$$

where $h^{(k)}$ is of degree strictly less than $n-k$.

Now, suppose that $\lambda_0, \ldots, \lambda_n$ are scalars such that

$$\lambda_0 f + \lambda_1 f' + \cdots + \lambda_n f^{(n)} = 0. \tag{10}$$

In order for the polynomial on the left hand side to be zero each coefficient must be zero. The leading coefficient (coefficient of the highest power of x) is the x^n coefficient which is $\lambda_0 a$. For this to be zero we see that $\lambda_0 = 0$ (since $a \neq 0$ by assumption). Thus, the equation simplifies to

$$\lambda_1 f' + \lambda_2 f'' + \cdots \lambda_n f^{(n)} = 0. \tag{11}$$

This time, after our simplification, the leading coefficient of the polynomial on the left hand side is x^{n-1} and it is read off as $\lambda_1 na$. Again, this is only zero when $\lambda_1 = 0$. Thus, our equation simplifies to

$$\lambda_2 f'' + \lambda_3 f''' + \cdots + \lambda_n f^{(n)} = 0. \tag{12}$$

Continuing we see that each coefficient must be zero. Thus, the set is linearly indepenent. Since it is of size $n+1 = \dim \mathsf{P}_n$ it must be a basis.

Question 29

I will state part (a) as a theorem.

Theorem 0.2. *Let* $\mathsf{W}_1, \mathsf{W}_2$ *be subspaces of a vector space* V. *Then*

$$\dim(\mathsf{W}_1 + \mathsf{W}_2) = \dim \mathsf{W}_1 + \dim \mathsf{W}_2 - \dim(\mathsf{W}_1 \cap \mathsf{W}_2). \tag{13}$$

Proof. Start with a basis $\{u_1, \ldots, u_k\}$ for the intersection $\mathsf{W}_1 \cap \mathsf{W}_2$. By corollary 2 of §1.6, we know that we can extend this to a basis of W_1 call this $\{u_1, \ldots, u_k, v_1, \ldots, v_r\}$. Similarly, applying the same corollary, extend the original set to a basis for W_2 as well; call this $\{u_1, \ldots, u_k, w_1, \ldots, w_s\}$. Notice that the combined set

$$S = \{u_1, \ldots, u_k, v_1, \ldots, v_r, w_1, \ldots, w_s\} \tag{14}$$

is of size $k+r+s$. Also, this combined set certainly generates $\mathsf{W}_1 + \mathsf{W}_2$. Thus $\dim(\mathsf{W}_1 + \mathsf{W}_2) \leq k + r + s$. On the other hand, $\dim \mathsf{W}_1 = k + r$, $\dim \mathsf{W}_2 = k + s$. Thus

$$\dim \mathsf{W}_1 + \dim \mathsf{W}_2 - \dim(\mathsf{W}_1 \cap \mathsf{W}_2) = (k+r) + (k+s) - k = k + r + s.$$

Thus, we have shown the $\leq$ part of the theorem.

We now finish by showing $\geq$ and hence the theorem follows. We do this by showing that the combined set in equation (14) is linearly independent hence a basis for $\mathsf{W}_1 + \mathsf{W}_2$. Indeed, assume that

$$\mathbf{0} = a_1 u_1 + \cdots + a_k u_k + b_1 v_1 + \cdots + b_r v_r + c_1 w_1 + \cdots + c_s w_s \tag{15}$$

for scalars, a_i, b_j, c_k. Notice that this equation is of the form

$$\mathbf{0} = \mathbf{u} + \mathbf{v} + \mathbf{w} \tag{16}$$

for vectors $\mathbf{u} = a_1 u_1 + \cdots + a_k u_k \in W_1 \cap W_2$, $\mathbf{v} = b_1 v_1 + \cdots + b_r v_r \in W_1$, and $\mathbf{w} = c_1 w_1 + \cdots + c_s w_s \in W_2$. Now, we can rearrange this equation to write it as

$$-\mathbf{w} = \mathbf{u} + \mathbf{v}. \tag{17}$$

Since the left hand side manifestly lives in W_2 while the right hand side manifestly lives in W_1 we see that $\mathbf{w} \in W_1 \cap W_2$. Thus, we can find scalars $\lambda_1, \ldots, \lambda_k$ such that

$$-\mathbf{w} = \lambda_1 u_1 + \cdots + \lambda_k u_k. \tag{18}$$

On the other hand, $-\mathbf{w} = -(c_1 w_1 + \cdots + c_s w_s)$. Thus we have the equation

$$\mathbf{0} = \lambda_1 u_1 + \cdots + \lambda_k u_k + c_1 w_1 + \cdots + c_s w_s. \tag{19}$$

By construction, the set $\{u_1, \ldots, u_k, w_1, \ldots, w_s\}$ is linearly independent hence all scalars above must be zero. In particular, $c_1 = c_2 = \cdots = c_s = 0$. The original equation (15) reduces to

$$\mathbf{0} = a_1 u_1 + \cdots + a_k u_k + b_1 v_1 + \cdots + b_r v_r. \tag{20}$$

But, we know the set $\{u_1, \ldots, u_k, v_1, \ldots, v_r\}$ is also linearly independent. Hence $a_1 = a_2 = \cdots = a_k = b_1 = \cdots = b_r = 0$ as well. The theorem now follows. $\square$