Solutions to selected exercises from §2.1

0.1 Question 14

(a) Suppose that T carries linearly independent subsets to linearly independent subsets. Suppose that $x \in \mathsf{V}$ is not equal to zero. Then $\{v\}$ is linearly independent. By assumption, so is $\{\mathsf{T}(v)\}$. Therefore, $\mathsf{T}(v) \neq 0$. This shows that $\ker \mathsf{T} = \{0\}$ and hence T is injective. On the other hand, suppose that T is injective and let $\{u_1, \ldots, u_k\}$ be an arbitrary linearly independent subset. Suppose

$$\sum_i \lambda_i \mathsf{T}(u_i) = \mathbf{0}. \tag{1}$$

By linearity, this implies that

$$\mathsf{T}\left(\sum_i \lambda_i u_i\right) = \mathbf{0}. \tag{2}$$

Thus $\sum_i \lambda_i u_i = \mathbf{0}$. But, since $\{u_i\}$ is linearly independent this implies $\lambda_1 = \cdots = \lambda_k = 0$.

(b) Suppose S is linearly independent and suppose there is a collection of vectors $\{u_i\} \subset S$ such that $\sum \lambda_i \mathsf{T}(u_i) = 0$. This means $\mathsf{T}(\sum \lambda_i u_i) = 0$. But since T is injective this means that $\sum \lambda_i u_i = 0$ and hence $\lambda_i = 0$ for all i. This shows that $T(S)$ is linearly independent. Conversely, we can read this backwards.

(c) Part (a) implies $\mathsf{T}(\beta)$ is linearly independent. Since T is both injective and surjective we know that $\dim \mathsf{V} = \dim \mathsf{W}$. Thus, $\mathsf{T}(\beta)$ is a basis.

0.2 Question 17

The dimension theorem is

$$\dim \mathsf{V} = \dim \operatorname{Im} \mathsf{T} + \dim \ker \mathsf{T}. \tag{3}$$

(a) Suppose $\dim V < \dim \mathsf{W}$. Then, we can use the dimension theorem to conclude:

$$\dim \operatorname{Im} \mathsf{T} = \dim \mathsf{V} - \dim \ker \mathsf{T} < \dim \mathsf{W} - \dim \ker \mathsf{T} \leq \dim \mathsf{W}. \tag{4}$$

The final inequality uses the fact that $\dim \ker \mathsf{T} \geq 0$. THis shows $\dim \operatorname{Im} \mathsf{T} < \dim \mathsf{W}$ hence T cannot be onto.

(b) Similar.

0.3 Question 26

We will write elements $x \in \mathsf{W}_1 \oplus \mathsf{W}_2$ as $x = x_1 + x_2$.

(a) Suppose $\mathsf{T}(x) = x$. The left side is $\mathsf{T}(x_1 + x_2) = x_1$. The right hand side is $x_1 + x_2$. Thus, $x_2 = 0$. So $x = x_1 \in \mathsf{W}_1$. On the other hand if $x = x_1 \in \mathsf{W}_1$ then certainly $\mathsf{T}(x) = x$.

(b) Suppose $y \in \operatorname{Im} \mathsf{T}$. Then there is $x \in \mathsf{V}$ with $\mathsf{T}(x) = y$. But this means that $x_1 = y_1 + y_2$. But this implies $y_2 = 0$ since $x_1 - y_1 \in \mathsf{W}_1$ by the subspace property. Thus $y \in \mathsf{W}_1$. On the other hand, if $y \in \mathsf{W}_1$ then $y = \mathsf{T}(y)$ by part (a), so it is a value of T. This shows $\operatorname{Im} \mathsf{T} = \mathsf{W}_1$. Suppose $x \in \ker \mathsf{T}$. Thus $x_1 = 0$. So $x \in \mathsf{W}_2$. On the other hand, if $x = x_2 \in \mathsf{W}_2$ then $\mathsf{T}(x) = 0$.

(c) T is the identity transformation.

(d) T is the zero transformation.

0.4 Problem 27

(a) Let $\{u_i\}$ be a basis for W. We know we can extend it $\{u_i, v_j\}$ to a basis for V. Let $\mathsf{W}' = \operatorname{span}\{v_j\}$. We claim that $\mathsf{V} = \mathsf{W} \oplus \mathsf{W}'$. First, we show that $\mathsf{W} \cap \mathsf{W}' = \{0\}$. Suppose x is in this intersection. Then $x = \sum \lambda_i u_i$ and $x = \sum \mu_j v_j$ for some coefficients λ_i, μ_j. But this means that

$$\sum \lambda_i u_i - \sum \mu_j v_j = 0. \tag{5}$$

But, since $\{u_i, v_j\}$ is a basis we see that $\mu_i = 0, \lambda_j = 0$ for all i, j. Thus $x = 0$ and we showed $W \cap W' = \{0\}$.

(b) Let $\mathsf{V} = \mathbb{R}^2$. Let W be x-axis $\operatorname{span}\{e_1\}$. Let W′ be y-axis $\operatorname{span}\{e_2\}$. Let W″ be the line $\operatorname{span}\{e_1 + e_2\}$. The projections along each of these onto W have the desired properties.