Solutions to selected exercises from §2.2

Recall the very useful definitions.

Definition 0.1. Let V be a vector space of dimension n and suppose $\beta = \{v_i\}$ is an ordered basis for V. For any $v \in V$ define the column vector $[x]_\beta \in \mathbb{R}^n$ by the formula

$$[x]_\beta \stackrel{\text{def}}{=} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \tag{1}$$

where $v = \sum_i a_i v_i$.

Definition 0.2. Let $\mathsf{T}\colon \mathsf{V} \to \mathsf{W}$ be a linear transformation. Let $\beta = \{v_i\}$ be an ordered basis for V. Let $\gamma = \{w_j\}$ be an ordered basis for W. The matrix representation of T with respect to these bases is the matrix

$$[\mathsf{T}]_\beta^\gamma \stackrel{\text{def}}{=} \begin{bmatrix} a_1^1 & a_2^1 & \cdots & a_n^1 \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^m & a_2^m & \cdots & a_n^m \end{bmatrix}$$

where the entries $\{a_i^j\}$ are defined by

$$\mathsf{T}(v_i) = \sum_j a_i^j w_j. \tag{2}$$

Question 8

We will work with the field $\mathbb{F} = \mathbb{R}$ for familiarity. If β is a basis, define $\mathsf{T}\colon \mathsf{V} \to \mathbb{R}^n$ by $T(x) = [x]_\beta$. Suppose $x = \sum_i \lambda_i u_i$ and $y = \sum_j \mu_j u_j$. Then

$$x + y = \sum_i (\lambda_i + \mu_i) u_i \tag{3}$$

So, the ith row of the column vector $[x+y]_\beta$ is $\lambda_i + \mu_i$. But, this is the same as the ith row of the column vector $[x]_\beta + [y]_\beta$. Thus $[x+y]_\beta = [x]_\beta + [y]_\beta$. To show $[\lambda x]_\beta = \lambda [x]_\beta$ is similar.

Question 10

Let a_j^i denote the ij entry of $[T]_\beta$. That is, the entry in the ith row and jth column. Then a_j^i is the coefficient of v_j in $T(v_i)$. Since $T(v_i) = v_i + v_{i-1}$ this means that

$$a_j^i = \begin{cases} 1, & j = i \text{ or } i+1 \\ 0, & \text{else.} \end{cases} \tag{4}$$

Question 12

Definition 0.3. A square matrix $A = (a_i^j)$ is ***upper triangular*** if $a_i^j = 0$ for $i < j$.

Example. The 3×3 matrix

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & -4 \end{bmatrix}$$

is upper triangular.

Now onto the solution of the problem. Let $\beta = \{v_i\}$ be a basis for V. Then, the entries (a_i^j) of the matrix $[\mathsf{T}]_\beta[\mathsf{T}]_\beta^\beta$ are defined by

$$\mathsf{T}(v_i) = \sum_j a_i^j v_j. \tag{5}$$

Suppose that for each i that $\mathsf{T}(v_i)$ is a linear combination of $\{v_1, \ldots, v_i\}$. This is true if and only if $a_i^j = 0$ for $j > i$. Thus, the matrix $[\mathsf{T}]_\beta$ is uppertriangular if and only if this condition holds.

Question 17

Let $\{u_i\}$ be an ordered basis for $\ker \mathsf{T}$. Extend it to an ordered basis for V, call this $\beta = \{v_j, u_i\}$. We claim that $\{\mathsf{T}(v_j)\}$ is a basis for $\operatorname{Im} \mathsf{T}$. By the dimension theorem, it is of the right size. Also, it generates by theorem from class. We can extend $\{\mathsf{T}(v_j)\}$ to a basis for all of W, call this $\gamma = \{\mathsf{T}(v_j), w_k\}$. The entries a_i^j of $[\mathsf{T}]_\beta^\gamma$ are all zero unless $i = j \leq \dim \operatorname{Im} \mathsf{T}$. In particular, this matrix is diagonal.