Solutions to selected exercises from §2.4

Question 4

Think about tying and untying your shoes.

Question 7

(a) We prove by constradiction. Suppose there existed A^{-1}. Then from the equation $A^2 = 0$ we can multiply, on the left, by A^{-1} to arrive at the equation $A^{-1}A^2 = A^{-1} \cdot 0$. The right hand side is clearly zero and the left is A, so this implies $A = 0$.

(b) This is impossible. Since A is invertible, we can multiply $AB = 0$ on the left by A^{-1} to arrive at $B = 0$.

Question 10

Define $\mathsf{T}\colon \mathsf{V} \longrightarrow \mathbb{R}^3$ by the formula

$$\mathsf{T}\left(\begin{bmatrix} a & a+b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \tag{1}$$

Note that this map is well-defined! That is, if

$$\begin{bmatrix} a & a+b \\ 0 & c \end{bmatrix} = \begin{bmatrix} a' & a'+b' \\ 0 & c' \end{bmatrix} \tag{2}$$

Then this implies that $a = a', b = b', c = c'$.

Let's first check that this map is injective.

$$\mathsf{T}\left(\begin{bmatrix} a & a+b \\ 0 & c \end{bmatrix}\right) = \mathsf{T}\left(\begin{bmatrix} a' & a'+b' \\ 0 & c' \end{bmatrix}\right) \tag{3}$$

then this means that

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a' \\ b' \\ c' \end{bmatrix}. \tag{4}$$

So, certainly the original two 2×2 matrices are equal. Thus, the map is injective.

The map is surjective since the dimensions of the domain and codomain agree. So the map is an isomorphism.

Question 16

We first show that Φ is injective. Suppose $\Phi(A) = \Phi(A')$ where A, A' are $n \times n$ matrices. Then $B^{-1}AB = B^{-1}A'B$. Multiplying on the left by B and on the right by B^{-1} we conclude $A = A'$. Again, we apply the dimension theorem to conclude it is an isomorphism.

Question 24

We will cover this in class on March 16.

The main new input to this question is the *quotient space*, we set this up now.

Let's step back and consider the concept of an *equivalence relation* on a set S. This is a relation $\sim$ in the set S with the following three properties:

1. (Reflexive) $a \sim a$ for all $a \in S$.
2. (Symmetry) If $a \sim b$ then $b \sim a$.
3. (Transitive) If $a \sim b$ and $b \sim c$ then $a \sim c$.

The *equivalence class* associated to an element $a \in S$ is defined to be

$$[a] \stackrel{\text{def}}{=} \{x \in S \mid a \sim x\}. \tag{5}$$

Notice that if $a \sim b$ then $[a] = [b]$.

The set of equivalence classes is denoted

$$S/{\sim} \stackrel{\text{def}}{=} \{[a] \mid a \in S\}. \tag{6}$$

Notice that as a set $S/{\sim}$ is much smaller than S. Additionally, there is a canonical function $\pi \colon S \to S/{\sim}$ defined by $\pi(a) = [a]$.

Now, we turn back to vector spaces. Let V be a vector space and suppose that $\mathsf{W} \subset \mathsf{V}$ is a subspace. Define the following equivalence relation on V:

$$v \sim v' \iff v - v' \in \mathsf{W}. \tag{7}$$

Let's check this is an equivalence relation: (1) clearly reflexive since $0 \in \mathsf{W}$, (2) if $v - v' \in \mathsf{W}$ then $-(-v - v') = v' - v$ is in W as well, (3) if $v - v' \in \mathsf{W}$ and $v' - v'' \in \mathsf{W}$ then

$$v - v'' = (v - v') + (v' - v'') \in \mathsf{W} \tag{8}$$

so we also have transitivity.

We denote the set of equivalence classes by

$$\mathsf{V}/\mathsf{W} \stackrel{\text{def}}{=} \mathsf{V}/{\sim}\ . \tag{9}$$

We will show that V/W has the natural structure of a vector space so that the natural function

$$\pi \colon \mathsf{V} \to \mathsf{V}/\mathsf{W} \tag{10}$$

is a linear transformation.

It is convenient to use a slightly different notation for equivalence classes in the setting of linear algebra: we write, for $v \in \mathsf{V}$:

$$[v] = v + \mathsf{W}. \tag{11}$$

This also makes it easier to remember the subspace W that is being used to define the equivalence relation. Notice that if $w \in \mathsf{W}$ then

$$v + w + \mathsf{W} = v + \mathsf{W} \tag{12}$$

so the notation is sensible.

We define the addition of two equivalence classes in the natural way:

$$(v + \mathsf{W}) + (v' + \mathsf{W}) = (v + v') + \mathsf{W}. \tag{13}$$

And scalar multiplication:

$$\lambda \cdot (v + \mathsf{W}) = \lambda v + \mathsf{W}. \tag{14}$$

It is immediate to check that these operations endow V/W with a vector space structure. The zero vector is $0_V + \mathsf{W}$.

Theorem 0.1. *Suppose that* V *is* n*-dimensional and* $\mathsf{W} \subset \mathsf{V}$ *is an* m*-dimensional subspace. Then* V/W *is* $(n-m)$*-dimensional.*

Proof. This follows from the more basic fact that $\pi \colon \mathsf{V} \to \mathsf{V}/\mathsf{W}$ is surjective. Thus, by the dimension theorem we have that

$$n = \dim \ker \pi + \dim \mathsf{V}/\mathsf{W}. \tag{15}$$

Now, we claim that $w \in \ker \pi$ if and only if $w \in \mathsf{W}$. Indeed, if $\pi(w) = w + \mathsf{W} = 0 + \mathsf{W}$ then $w - 0 = w \in \mathsf{W}$ by the definition of the equivalence relation. It follows that $\ker \pi = \mathsf{W}$ and hence from the formula above

$$\dim \mathsf{V}/\mathsf{W} = n - m. \tag{16}$$

□

Now, we can finally turn to problem 24. We start with a *surjective* linear transformation $\mathsf{T} \colon \mathsf{V} \to \mathsf{Z}$. Then the problem defines

$$\overline{\mathsf{T}} \colon \mathsf{V}/\ker \mathsf{T} \to \mathsf{Z} \tag{17}$$

by the formula $\overline{\mathsf{T}}(v+\ker \mathsf{T}) = \mathsf{T}(v)$. This is well-defined: if $v+\ker \mathsf{T} = v'+\ker \mathsf{T}$ then $v-v' \in \ker \mathsf{T}$. Thus $\mathsf{T}(v) = \mathsf{T}(v')$. This is part (a). It is an easy exercise to show that $\overline{\mathsf{T}}$ is linear.

We now do part (c), to show that T is an isomorphism. Let $n = \dim \mathsf{V}$ and $k = \dim \ker \mathsf{T}$. Then, we know from the previous theorem

$$\dim \mathsf{V}/\ker \mathsf{T} = n - k. \tag{18}$$

On the other hand, by the dimension theorem, we know that

$$n = k + \dim \operatorname{Im} \mathsf{T}. \tag{19}$$

By assumption, T is surjective, so $\dim \operatorname{Im} \mathsf{T} = \dim \mathsf{Z}$. It follows that

$$\dim \mathsf{Z} = n - k = \dim \mathsf{V}/\ker \mathsf{T}. \tag{20}$$

Thus, in order to show that T is an isomorphism, we just need to show that it's surjective (or injective). But, $\overline{\mathsf{T}}$ is automatically surjective since T is!

In the diagram of part (d) the map "η" in the book is the one I am calling $\pi \colon \mathsf{V} \to \mathsf{V}/\ker \mathsf{T}$. Commutativity of this diagram is the statement that

$$\mathsf{T} = \overline{\mathsf{T}} \circ \pi. \tag{21}$$

To check this it suffices to check that when we plug in v to both sides, for any v, we get the same result. The left hand side is just $\mathsf{T}(v)$. The right hand side is $\overline{\mathsf{T}}(\pi(v)) = \overline{\mathsf{T}}(v+\ker \mathsf{T}) = \mathsf{T}(v)$.Thus, the diagram commutes. This concludes problem 24.