Solutions to selected exercises from §3.2

Question 3

One direction is obvious. On the other hand, suppose that $\text{rank}(A) = 0$. By the theorem we proved in class, this means that the number of linearly independent columns is zero. So, this is the zero matrix.

Question 5

$$\textbf{(a)} \quad A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$

To find the inverse, we will row reduce the augmented matrix $\left[\begin{array}{c|c} A & \mathbb{1} \end{array}\right]$ to an augmented matrix of the form $\left[\begin{array}{c|c} \mathbb{1} & A^{-1} \end{array}\right]$.

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - R_1} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -1 & -1 & 1 \end{array}\right] \xrightarrow{R_2 \leftarrow -R_2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 \end{array}\right]$$

$$\xrightarrow{R_1 \leftarrow R_1 - 2R_2} \left[\begin{array}{cc|cc} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \end{array}\right]$$

Hence

$$\text{rank}(A) = 2, \qquad A^{-1} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}.$$

$$\textbf{(b)} \quad B = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

Row reduce:

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$$

Hence

$$\text{rank}(B) = 1.$$

Since the rank is less than 2, B is not invertible.

$$\textbf{(c)} \quad C = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 4 \\ 2 & 3 & -1 \end{pmatrix}$$

Row reduce:

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 4 \\ 2 & 3 & -1 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - R_1,\ R_3 \leftarrow R_3 - 2R_1} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{pmatrix}$$

$$\xrightarrow{R_3 \leftarrow R_3 + R_2} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 \leftarrow R_1 - 2R_2} \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence

$$\operatorname{rank}(C) = 2.$$

Since the rank is less than 3, C is not invertible.

$$\textbf{(e)} \quad E = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$$

To find the inverse, row reduce $\begin{bmatrix} E & I \end{bmatrix}$:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 + R_1,\ R_3 \leftarrow R_3 - R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 3 & 3 & 1 & 1 & 0 \\ 0 & -2 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow \frac{1}{3} R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & -2 & 0 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 + 2R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 2 & -\frac{1}{3} & \frac{2}{3} & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow \frac{1}{2} R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 & -\frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - R_3,\ R_1 \leftarrow R_1 - R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & \frac{7}{6} & -\frac{1}{3} & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{array}\right]$$

$$\xrightarrow{R_1 \leftarrow R_1 - 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{6} & -\frac{1}{3} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{array}\right]$$

Hence

$$\operatorname{rank}(E) = 3, \qquad E^{-1} = \begin{pmatrix} \frac{1}{6} & -\frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{pmatrix}.$$

$$\textbf{(f)} \quad F = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Row reduce:

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - R_1,\ R_3 \leftarrow R_3 - R_1} \begin{pmatrix} 1 & 2 & 1 \\ 0 & -2 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\xrightarrow{R_2 \leftarrow -\frac{1}{2} R_2} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix} \xrightarrow{R_3 \leftarrow R_3 + R_2} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 \leftarrow R_1 - 2R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence

$$\operatorname{rank}(F) = 2.$$

Since the rank is less than 3, F is not invertible.

Question 11

Suppose that the rank of the big matrix B is r. Let us denote the small matrix B' by

$$B' = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix} \tag{1}$$

for some vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^m$. Thus, we can write the big matrix in the form

$$B = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \mathbf{0} & \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix} \tag{2}$$

where $\mathbf{0} \in \mathbb{R}^m$ is the zero vector. Observe that

$$\begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix} \notin \operatorname{span}\left\{ \begin{bmatrix} 0 \\ \mathbf{v}_1 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ \mathbf{v}_n \end{bmatrix} \right\}. \tag{3}$$

Next, take a maximally linearly independent set $S \subset \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. Then, we know that

$$S \cup \{ \begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix} \}$$

is still linearly independent. Since

$$\# \left(S \cup \{ \begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix} \} \right) = r, \tag{4}$$

by assumption, this implies $\#S = r - 1$. Thus the rank of B' is $r - 1$.

Question 14

(a) Suppose that $y \in \operatorname{Im}(\mathsf{T} + \mathsf{U})$. This means that there exists $x \in \mathsf{V}$ such that

$$y = (\mathsf{T} + \mathsf{U})(x) = \mathsf{T}(x) + \mathsf{U}(x). \tag{5}$$

The second equality is the definition of the sum of two linear transformations. But, by definition of the sum of subspaces

$$\mathsf{T}(x) + \mathsf{U}(x) \in \operatorname{Im} \mathsf{T} + \operatorname{Im} \mathsf{U}. \tag{6}$$

(b) If W is finite-dimensional then the ranks (dimension of the image) of T, U and hence $\mathsf{T} + \mathsf{U}$ are all finite. This now follows from part (a).

(c) This is a formal consequence of (a) and (b) where we apply it to the case that $\mathsf{T} = \mathsf{L}_A$, $\mathsf{U} = \mathsf{L}_B$.

Question 18

Let $\mathbf{a}_j$ be the jth column of A. Then, for each j, $\mathbf{a}_j$ can be thought of as a $m \times 1$ matrix. Let $\mathbf{b}^i$ be the ith row of B. Then, for each i, $\mathbf{b}^i$ can be thought of as a $1 \times p$ matrix.

In particular $\mathbf{a}_j \mathbf{b}^i$ is an $m \times p$ matrix. Moreover, from the formula of matrix multiplication we have that

$$AB = \sum_{k=1}^{n} \mathbf{a}_k \mathbf{b}^k. \tag{7}$$

Now, since $\operatorname{rank}(\mathbf{a}_k) \leq \min\{m, 1\} \leq 1$ and $\operatorname{rank}(\mathbf{b}^k) \leq \min\{1, n\}$, we see that $\operatorname{rank}(\mathbf{a}_k \mathbf{b}^k) \leq 1$ as well. This proves the result.

Question 21

The left multiplication of A is a linear transformation $\mathsf{L}_A \colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$. We are assuming that $\dim \operatorname{Im} \mathsf{L}_A = m$. In other words, L_A is surjective. In particular, for each standard basis vector $e_j \in \mathbb{R}^m$ we can find a vector $v_j \in \mathbb{R}^n$ such that $\mathsf{L}_A(v_j) = Av_j = e_j$.

Define a linear transformation $\mathsf{S} \colon \mathbb{R}^m \longrightarrow \mathbb{R}^n$ by the formula $\mathsf{S}(e_j) = v_j$ for $j = 1, \ldots, m$. Notice that

$$(\mathsf{L}_A \circ \mathsf{S})(e_j) = \mathsf{L}_A(v_j) = e_j. \tag{8}$$

Thus $\mathsf{L}_A \circ \mathsf{S} = \mathbb{1}_{\mathbb{R}^m}$. Let $B = [\mathsf{S}]_\beta^\gamma$ where β, γ are the standard ordered bases. Then $AB = \mathbb{1}_{m \times m}$.

Question 22

Left multiplication is a linear transformation $\mathsf{L}_B \colon \mathbb{R}^m \longrightarrow \mathbb{R}^n$. Since $\dim \operatorname{Im} \mathsf{L}_B = m$, by the dimension theorem we conclude that L_B is injective. This implies that if $\beta = \{e_1, \ldots, e_m\}$ is the standard basis for $\mathbb{R}^m$ then

$$\{Be_1, \ldots, Be_m\} \subset \mathbb{R}^n \tag{9}$$

is a linearly independent subset. We can complete this linearly independent subset to a basis

$$\{Be_1, \ldots, Be_m, u_1, \ldots, u_k\} \tag{10}$$

for some vectors $u_1, \ldots, u_k$.

Using this basis we can define a linear transformation $\mathsf{S} \colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ by the formulas

$$\mathsf{S}(Be_j) = e_j, \quad \mathsf{S}(u_i) = 0 \tag{11}$$

for $j = 1, \ldots, m$ and $i = 1, \ldots, k$. Finally, let $A = [\mathsf{S}]_\beta^\gamma$ where β, γ are the standard ordered bases. Then $AB = \mathbb{1}$ similarly to the above question.