Solutions to selected exercises from §4.2

Question 23

Suppose that

$$A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}$$

is an $n \times n$ upper triangular matrix. Then

$$\mathbf{a}_1 = a_1^1 e_1 \quad (1)$$
$$\mathbf{a}_2 = a_2^2 e_2 + a_2^1 e_1 \quad (2)$$
$$\vdots \qquad \vdots \quad (3)$$
$$\mathbf{a}_n = a_n^n e_n + \cdots + a_n^1 e_1. \quad (4)$$

We use multi-linearity and alternating property to simplify:

$$\det(A) = \det(\mathbf{a}_1, \ldots, \mathbf{a}_n) \quad (5)$$
$$= \det(a_1^1 e_1, \mathbf{a}_2, \ldots, \mathbf{a}_n) \quad (6)$$
$$= a_1^1 \det(e_1, a_2^2 e_2 + a_2^1 e_1, \mathbf{a}_3, \ldots, \mathbf{a}_n) \quad (7)$$
$$= a_1^1 a_2^2 \det(e_1, e_2, \mathbf{a}_3, \cdots, \mathbf{a}_n). \quad (8)$$

Here, the first line in the definition. The second line expands $\mathbf{a}_1$ as above. The third line uses n-linearity of the determinant. The fourth line uses the alternating property of the determinant, as well as the n-linearity property. The key point in the fourth line is that the term

$$\det(e_1, a_2^1 e_1, \mathbf{a}_3, \ldots, \mathbf{a}_n) = 0 \quad (9)$$

since there is a repeated basis vector. We continue in this fashion to obtain

$$\det(A) = a_1^1 \cdots a_n^n \det(e_1, \ldots, e_n) = a_1^1 \cdots a_n^n \quad (10)$$

Question 25

If $A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}$ where $\mathbf{a}_i$ are the column vectors, then

$$kA = \begin{bmatrix} k\mathbf{a}_1 & \cdots & k\mathbf{a}_n \end{bmatrix}. \quad (11)$$

We now use n-linearity n times:

$$\det(kA) = \det(k\mathbf{a}_1, \ldots, k\mathbf{a}_n) \quad (12)$$
$$= k \det(\mathbf{a}_1, k\mathbf{a}_2, \ldots, k\mathbf{a}_n) \quad (13)$$
$$\vdots \quad (14)$$
$$= k^n \det(\mathbf{a}_1, \ldots, \mathbf{a}_n) \quad (15)$$
$$= k^n \det(A). \quad (16)$$