Solutions to selected exercises from §4.3

Question 10

If M is nilpotent then there exists an integer k such that $M^k = 0$. Take the determinant of both sides of this equation to get $\det(M^k) = 0$. On the other hand, we showed that $\det(AB) = (\det A)(\det B)$. So that $\det(M^k) = \det(M)^k$. In other words $\det(M^k) = \det(M)^k = 0$. This implies that $\det(M) = 0$.

Question 11

Suppose $M^t = -M$ and M is an $n \times n$ matrix where n is *odd*. Take the determinant of both sides to get

$$\det(M^t) = \det(-M). \tag{1}$$

In class, we showed that $\det(M^t) = \det(M)$. On the other hand $\det(-M) = (-1)^n \det(M)$ since det is an n-linear function of the column vectors of M. Since n is odd this implies that $\det(M) = -\det(M)$. But, this implies $\det(M) = 0$ so that M is not invertible.

If n is even, a skew-symmetric matrix can be either invertible or noninvertible. For an invertible example, consider

$$M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{2}$$

For a noninvertible example when n is even, simply take $M = 0$.

Question 12

$$\det(QQ^t) = \det(Q)\det(Q^t) = \det(Q)\det(Q) = \det(Q)^2 = 1$$

Question 15

Two matrices A, B are **similar** if there exists a matrix P such that

$$B = P^{-1}AP. \tag{3}$$

Take the determinant of both sides:

$$\det(B) = \det(P^{-1}AP) = \det(P^{-1})\det(A)\det(P) = \det(P)^{-1}\det(A)\det(P) = \det A. \tag{4}$$

Question 24

We will prove that

$$\det(A + t\mathbb{1}) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0 \tag{5}$$

using induction.

Let

$$B = (A + t\mathbb{1})^t. \tag{6}$$

Notice that transpose does not change the determinant, so it suffices to compute the determinant of B. Write the columns of B as

$$B = \begin{bmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_n \end{bmatrix}. \tag{7}$$

Then, for example

$$\mathbf{b}_1 = te_1 + a_0 e_n \tag{8}$$

By multilinearity, we therefore have

$$\det(B) = t\det(e_1, \mathbf{b}_2, \ldots, \mathbf{b}_n) + a_0 \det(e_n, \mathbf{b}_2, \ldots, \mathbf{b}_n) \tag{9}$$

Focus on the second term, the one which multiplies a_0. Applying alternating property we get

$$(-1)^{n-1} \det(\mathbf{b}_2, \cdots, \mathbf{b}_{n-1}, e_n) \tag{10}$$

where now $\begin{bmatrix} \mathbf{b}_2 & \cdots & \mathbf{b}_{n-1} & e_n \end{bmatrix}$ is an uppertriangular matrix with $(n-1)$ -1's along the diagonal and a single 1, therefore its determinant is $(-1)^{n-1}$. The two copies of $(-1)^{n-1}$ cancel, and therefore we see that this determinant is 1. Thus

$$\det B = t\det(e_1, \mathbf{b}_2, \ldots, \mathbf{b}_n) + a_0. \tag{11}$$

It remains to analyze the first term on the right hand side. For $j > 1$, let $\tilde{\mathbf{b}}_j \in \mathbb{R}^{n-1}$ be the projection of $\mathbf{b}_j$ onto its last $(n-1)$ components. Then

$$\det(e_1, \mathbf{b}_2, \ldots, \mathbf{b}_n) = t\det(\tilde{\mathbf{b}}_2, \ldots, \tilde{\mathbf{b}}_n)$$

Observe, that the $(n-1) \times (n-1)$ matrix is of the same type as the original matrix B, except where the last row is labeled $a_1, \ldots, a_n$. By the inductive hypothesis, we conclude that

$$\det(\tilde{\mathbf{b}}_2, \ldots, \tilde{\mathbf{b}}_n) = t^{n-1} + a_{n-1}t^{n-2} + \cdots + a_2 t + a_1. \tag{12}$$

Thus, in total we have

$$\det B = t(t^{n-1} + a_{n-1}t^{n-2} + \cdots + a_2 t + a_1) + a_0 = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0. \tag{13}$$