Solutions to selected exercises from §5.1

Question 4

(a) In the standard basis T is represented by the matrix

$$A = \begin{bmatrix} -2 & 3 \\ -10 & 9 \end{bmatrix}. \tag{1}$$

The characteristic polynomial of A is

$$\chi_A(t) = \det \begin{bmatrix} -2-\lambda & 3 \\ -10 & 9-\lambda \end{bmatrix} = \lambda^2 - 7\lambda + 12 = (\lambda - 3)(\lambda - 4). \tag{2}$$

So, there are two distinct eigenvalues: $\lambda_1 = 3, \lambda_2 = 4$.

For eigenvalue $\lambda_1 = 3$ we have

$$A - 3\mathbb{1} = \begin{bmatrix} -5 & 3 \\ -10 & 6 \end{bmatrix} \tag{3}$$

By a single elementary row operation we find that the kernel of this matrix is one-dimensional spanned by the eigenvector

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \tag{4}$$

For eigenvalue $\lambda_2 = 4$ we have

$$A - 4\mathbb{1} = \begin{bmatrix} -6 & 3 \\ -10 & 5 \end{bmatrix} \tag{5}$$

Again, by a single row operation we find that the kernel of this matrix is one-dimensional spanned by the eigenvector

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{6}$$

In the ordered basis $\beta = \{\mathbf{v}_1, \mathbf{v}_2\}$ the transformation T is represented by the diagonal matrix

$$\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}. \tag{7}$$

(b) In the standard basis the transformation T is represented by the matrix

$$\begin{bmatrix} 7 & -4 & 10 \\ 4 & -3 & 8 \\ -2 & b & -2 \end{bmatrix} \tag{8}$$

We compute the characteristic polynomial.

$$\begin{aligned}
\det(A - \lambda \mathbb{1}) &= \det \begin{bmatrix} 7-\lambda & -4 & 10 \\ 4 & -3-\lambda & 8 \\ -2 & 1 & -2-\lambda \end{bmatrix} && (9) \\
&= \det \begin{bmatrix} 7-\lambda & -4 & 10 \\ 0 & -1-\lambda & 4-2\lambda \\ -2 & 1 & -2-\lambda \end{bmatrix} && (10) \\
&= (7-\lambda)\det \begin{bmatrix} -1-\lambda & 4-2\lambda \\ 1 & -2-\lambda \end{bmatrix} - 2\det \begin{bmatrix} -4 & 10 \\ -1-\lambda & 4-2\lambda \end{bmatrix} && (11) \\
&= -\lambda^3 + 2\lambda^2 + \lambda - 2. && (12)
\end{aligned}$$

Observe that $\lambda = \pm 1$ are roots of this polynomial since $\mp 1 + 2 \pm 1 - 2 = 0$. Dividing by $\lambda^2 - 1$ we see that the remaining root is $\lambda = 2$. Thus we have three distinct eigenvalues $\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = 2$.

To find the eigenvector for $\lambda = 1$ we compute the kernel of

$$A - \mathbb{1} = \begin{bmatrix} 6 & -4 & 10 \\ 4 & -4 & 8 \\ -2 & 1 & -3 \end{bmatrix} \tag{13}$$

Row reduce:

$$\begin{bmatrix} 6 & -4 & 10 \\ 4 & -4 & 8 \\ -2 & 1 & -3 \end{bmatrix} \xrightarrow{R_2 \leftarrow 3R_2 - 2R_1} \begin{bmatrix} 6 & -4 & 10 \\ 0 & -4 & 4 \\ -2 & 1 & -3 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftarrow 3R_3 + R_1} \begin{bmatrix} 6 & -4 & 10 \\ 0 & -4 & 4 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - 4R_3} \begin{bmatrix} 6 & -4 & 10 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

From the last row:

$$-x_2 + x_3 = 0 \implies x_2 = x_3.$$

From the first row:

$$6x_1 - 4x_2 + 10x_3 = 0 \implies 6x_1 + 6x_3 = 0 \implies x_1 = -x_3.$$

Therefore,

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

is an eigenvector for $\lambda_1 = 1$.

Next, $\lambda_2 = -1$

$$A + \mathbb{1} = \begin{bmatrix} 8 & -4 & 10 \\ 4 & -2 & 8 \\ -2 & 1 & -1 \end{bmatrix}$$

Row reduce:

$$\begin{bmatrix} 8 & -4 & 10 \\ 4 & -2 & 8 \\ -2 & 1 & -1 \end{bmatrix} \xrightarrow{R_2 \leftarrow 2R_2 - R_1} \begin{bmatrix} 8 & -4 & 10 \\ 0 & 0 & 6 \\ -2 & 1 & -1 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftarrow 4R_3 + R_1} \begin{bmatrix} 8 & -4 & 10 \\ 0 & 0 & 6 \\ 0 & 0 & 6 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 - R_2} \begin{bmatrix} 8 & -4 & 10 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

From the second row:

$$6x_3 = 0 \Rightarrow x_3 = 0.$$

From the first row:

$$8x_1 - 4x_2 = 0 \Rightarrow 2x_1 - x_2 = 0 \Rightarrow x_2 = 2x_1.$$

Therefore

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

is an eigenvector for $\lambda_2 = -1$.

Finally, we consider $\lambda_3 = 2$.

$$A - 2\mathbb{1} = \begin{bmatrix} 5 & -4 & 10 \\ 4 & -5 & 8 \\ -2 & 1 & -4 \end{bmatrix}$$

Row reduce:

$$\begin{bmatrix} 5 & -4 & 10 \\ 4 & -5 & 8 \\ -2 & 1 & -4 \end{bmatrix} \xrightarrow{R_2 \leftarrow 5R_2 - 4R_1} \begin{bmatrix} 5 & -4 & 10 \\ 0 & -9 & 0 \\ -2 & 1 & -4 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftarrow 5R_3 + 2R_1} \begin{bmatrix} 5 & -4 & 10 \\ 0 & -9 & 0 \\ 0 & -3 & 0 \end{bmatrix} \xrightarrow{R_3 \leftarrow 3R_3 - R_2} \begin{bmatrix} 5 & -4 & 10 \\ 0 & -9 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

From the second row:

$$-9x_2 = 0 \Rightarrow x_2 = 0.$$

From the first row:

$$5x_1 + 10x_3 = 0 \Rightarrow x_1 = -2x_3.$$

Therefore

$$\mathbf{v}_3 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

is an eigenvector for $\lambda_3 = 2$.

We conclude that with respect to the ordered basis $\beta = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ the transformation T is represented by the diagonal matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \tag{14}$$

(c) In the standard basis the transformation T is represented by the matrix

$$A = \begin{bmatrix} -4 & 3 & -6 \\ 6 & -7 & 12 \\ 6 & -6 & 11 \end{bmatrix},$$

the characteristic polynomial is

$$\chi_A(\lambda) = \det(A - \lambda I) = (\lambda - 2)(\lambda + 1)^2.$$

(Please comput it yourself for practice!) So the eigenvalues are

$$\lambda_1 = 2, \qquad \lambda_2 = -1.$$

Namely, there are only two distinct eigenvalues!

We now compute the eigenvectors by row reducing $(A - \lambda I)\mathbf{x} = 0$. That is, by describing the kernel of $A - \lambda \mathbb{1}$ for $\lambda = 2, -1$.

For $\lambda_1 = 2$ we have:

$$A - 2\mathbb{1} = \begin{bmatrix} -6 & 3 & -6 \\ 6 & -9 & 12 \\ 6 & -6 & 9 \end{bmatrix}$$

Row reduce:

$$\begin{bmatrix} -6 & 3 & -6 \\ 6 & -9 & 12 \\ 6 & -6 & 9 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 + R_1} \begin{bmatrix} -6 & 3 & -6 \\ 0 & -6 & 6 \\ 6 & -6 & 9 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftarrow R_3 + R_1} \begin{bmatrix} -6 & 3 & -6 \\ 0 & -6 & 6 \\ 0 & -3 & 3 \end{bmatrix} \xrightarrow{R_3 \leftarrow 2R_3 - R_2} \begin{bmatrix} -6 & 3 & -6 \\ 0 & -6 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

We see that the kernel is one-dimensional. From the second row:

$$-6x_2 + 6x_3 = 0 \Longrightarrow x_2 = x_3.$$

From the first row:

$$-6x_1 + 3x_2 - 6x_3 = 0.$$

Using $x_2 = x_3$,

$$-6x_1 - 3x_3 = 0 \Longrightarrow 2x_1 + x_3 = 0 \Longrightarrow x_1 = -\frac{1}{2}x_3.$$

Therefore

$$\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$$

is an eigenvector for $\lambda_1 = 2$.

Now, for $\lambda_2 = -1$:

$$A + \mathbb{1} = \begin{bmatrix} -3 & 3 & -6 \\ 6 & -6 & 12 \\ 6 & -6 & 12 \end{bmatrix}$$

Row reduce:

$$\begin{bmatrix} -3 & 3 & -6 \\ 6 & -6 & 12 \\ 6 & -6 & 12 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 + 2R_1} \begin{bmatrix} -3 & 3 & -6 \\ 0 & 0 & 0 \\ 6 & -6 & 12 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftarrow R_3 + 2R_1} \begin{bmatrix} -3 & 3 & -6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, the kernel is two-dimensional. There will be two linearly independent eigenvectors! The system is just

$$-3x_1 + 3x_2 - 6x_3 = 0$$

or

$$x_1 = x_2 - 2x_3.$$

Let $x_2 = s, x_3 = t$. Then we have shown a general vector in the kernel of $A + \mathbb{1}$ is of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}.$$

Thus, we have two linearly independent eigevectors for λ_2:

$$\{\mathbf{v}_2, \mathbf{v}_3\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

We conclude that T is diagonalizable, and with respect to the basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ the matrix representation of T is

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Question 9

Let M be an upper triangular matrix. Then notice that $M - t\mathbb{1}$ is still upper triangular for an scalar t. Thus, its determinant is a product of its diagonal elements. If $\lambda_1, \ldots, \lambda_n$ denote the diagonal elements of M, then we conclude the characteristic polynomial for M is

$$\chi_M(t) = (\lambda_1 - t) \cdots (\lambda_n - t). \tag{15}$$

This proves that the eigenvalues of M are exactly the diagonal elements of M.

Question 16

(a) Two matrices A, B are similar if there exists a matrix Q such that $B = Q^{-1}AQ$. Recall that for any two matrices A, B we have $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ (you should remember how to prove this, but it also appeared as an exercise from §2.3). Thus

$$\mathrm{tr}(B) = \mathrm{tr}(Q^{-1}AQ) = \mathrm{tr}(QQ^{-1}A) = \mathrm{tr}(A). \tag{16}$$

(b) For $\mathsf{T} \colon \mathsf{V} \longrightarrow \mathsf{V}$ define

$$\mathrm{tr}(\mathsf{T}) \stackrel{\text{def}}{=} \mathrm{tr}\left([\mathsf{T}]_\beta\right) \tag{17}$$

where β is any ordered basis of V. To see that this is well-defined, note that if γ is any other basis, then

$$[\mathsf{T}]_\gamma = [\mathbb{1}]_\gamma^\beta [\mathsf{T}]_\beta [\mathbb{1}]_\beta^\gamma = ([\mathbb{1}]_\beta^\gamma)^{-1} [\mathsf{T}]_\beta [\mathbb{1}]_\beta^\gamma. \tag{18}$$

Thus, take $Q = [\mathbb{1}]_\beta^\gamma$ and apply part (a) to see this is well-defined. (What we have shown is that the trace is indepdendent of the choice of basis.)