

Midterm 1 Review

This brief reference collects the notation and ideas used in the problems. Dots denote arbitrary-parameter derivatives; primes denote arclength derivatives.

Curves: speed, arclength, and curvature. A curve $c(t)$, $a \leq t \leq b$, is *regular* when $\dot{c}(t) \neq 0$. Its speed is $v(t) = \|\dot{c}(t)\|$, and arclength satisfies

$$\frac{ds}{dt} = v(t), \quad \frac{d}{ds} = \frac{1}{v(t)} \frac{d}{dt}, \quad \text{Length}(c) = \int_a^b v(t) dt.$$

In an arclength parametrization, $\|c'(s)\| = 1$. The unit tangent is $T = c'$, and ordinary curvature $\kappa = \|T'\| \geq 0$ measures how quickly the tangent turns per unit length. Where $\kappa > 0$, the Frenet frame in $\mathbb{R}^3$ is

$$T, \quad N = \frac{T'}{\kappa}, \quad B = T \times N.$$

Torsion measures the turning of the binormal; our sign convention is

$$T' = \kappa N, \quad N' = -\kappa T + \tau B, \quad B' = -\tau N.$$

For plane curves, fix $J(x, y) = (-y, x)$. The *signed* curvature k is defined by $T' = kJT$, so $\kappa = |k|$. Writing $T = (\cos \theta, \sin \theta)$ gives $\theta' = k$. To reconstruct a plane curve, integrate k to obtain θ , then integrate T to obtain c , using the initial conditions.

Surface patches, tangent vectors, and normals. A regular patch $f : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ has linearly independent vectors f_u, f_v . With $u_1 = u, u_2 = v$, the notation $f_i = \partial f / \partial u_i$ means a derivative of the *whole vector-valued map*, not its i -th component; $f_{ij} = \partial^2 f / \partial u_i \partial u_j$. At $p = f(u, v)$,

$$T_p S = \text{span}\{f_u, f_v\}, \quad \nu = \frac{f_u \times f_v}{\|f_u \times f_v\|}.$$

Either ν or $-\nu$ can be chosen as unit normal. A tangent vector $X = a^1 f_u + a^2 f_v$ has *coordinate column* $a = (a^1, a^2)^T$; $a \in \mathbb{R}^2$ and $X \in \mathbb{R}^3$ are different objects. For a regular level surface $S = F^{-1}(0)$,

$$T_p S = \ker dF_p = \nabla F(p)^\perp.$$

The tangent space consists of vectors; the affine tangent plane is $p + T_p S$.

The first fundamental form and its matrix g . The first fundamental form is the usual inner product restricted to the tangent plane: $I(X, Y) = \langle X, Y \rangle$. Its matrix in the coordinate basis (f_u, f_v) is

$$g = (g_{ij}) = \begin{pmatrix} \langle f_u, f_u \rangle & \langle f_u, f_v \rangle \\ \langle f_v, f_u \rangle & \langle f_v, f_v \rangle \end{pmatrix}.$$

Thus g converts coordinate columns into geometric lengths and angles. If X, Y have coordinate columns a, b , then

$$I(X, Y) = a^\top g b, \quad \|X\|^2 = a^\top g a, \quad \cos \angle(X, Y) = \frac{a^\top g b}{\sqrt{a^\top g a} \sqrt{b^\top g b}},$$

where the angle formula assumes nonzero vectors. The matrix g is symmetric and positive definite; it is the identity only when the chosen tangent basis is orthonormal. The area element is $dA = \sqrt{\det g} \, du \, dv$.

The second fundamental form and its matrix h . After choosing a unit normal ν , set

$$h = (h_{ij}) = \begin{pmatrix} \langle f_{uu}, \nu \rangle & \langle f_{uv}, \nu \rangle \\ \langle f_{vu}, \nu \rangle & \langle f_{vv}, \nu \rangle \end{pmatrix}, \quad II(X, Y) = a^\top h b.$$

The entries of h record the normal components of the second derivatives of the patch: they measure how the surface bends. The matrix h is symmetric since $f_{uv} = f_{vu}$, but unlike g , it need not be positive definite. Both matrices depend on the point and the chosen coordinates. For a unit-speed surface curve γ with $\gamma'(0) = X$, $II(X, X) = \langle \gamma''(0), \nu \rangle$, its signed normal curvature at that point.

The shape operator and principal curvatures. The shape operator $L : T_p S \rightarrow T_p S$ is minus the derivative of the unit normal along the surface. In coordinates,

$$L(f_i) = -\nu_i, \quad II(X, Y) = I(LX, Y), \quad [L] = g^{-1}h.$$

Its eigenvalues are the principal curvatures. A coordinate eigenvector a represents the principal direction $X = a^1 f_u + a^2 f_v$; normalize by $\sqrt{a^\top g a}$, not generally by $\|a\|$. Diagonalize $g^{-1}h$, not h alone.

The operator is self-adjoint for I , meaning $[L]^\top g = g[L]$. Its coordinate matrix need not be symmetric in a nonorthonormal basis. Reversing ν changes the signs of h , L , and the principal curvatures, but leaves g and area unchanged.

Try each question before reading its solution. This is a review set, not a timed exam. Questions are in black and solutions are in muted violet.

1. Recovering a curve from its tangent.

A unit-speed curve c satisfies $c(0) = 0$ and

$$T(s) = \left(\frac{4}{5} \cos s, \frac{4}{5} \sin s, \frac{3}{5} \right).$$

Find $c(s)$ and its curvature. Is its Frenet frame defined everywhere?

Solution. Since $c' = T$, integrate each component from 0 to s :

$$c(s) = \left(\frac{4}{5} \sin s, \frac{4}{5} (1 - \cos s), \frac{3}{5} s \right).$$

The given tangent has norm 1, so s really is arclength. Also,

$$T'(s) = \left(-\frac{4}{5} \sin s, \frac{4}{5} \cos s, 0 \right), \quad \kappa(s) = \|T'(s)\| = \frac{4}{5}.$$

Since $\kappa > 0$, the Frenet frame is defined for every s .

2. Torsion without a determinant.

For the curve in Problem 1, compute N , B , and τ , using

$$N = \frac{T'}{\kappa}, \quad B = T \times N, \quad B' = -\tau N.$$

Is this curve planar?

Solution. The normal and binormal are

$$N = (-\sin s, \cos s, 0), \quad B = \left(-\frac{3}{5} \cos s, -\frac{3}{5} \sin s, \frac{4}{5} \right).$$

Thus

$$B' = \left(\frac{3}{5} \sin s, -\frac{3}{5} \cos s, 0 \right) = -\frac{3}{5} N, \quad \tau = \frac{3}{5}.$$

The curve is not planar. For a planar Frenet curve, T and N lie in a fixed plane, so B is constant and $\tau = 0$.

3. Changing speed is not changing curvature.

Let c be the curve from Problem 1 and put $\alpha(t) = c(t^2)$ for $1 \leq t \leq 2$. Find its speed, length, curvature, and torsion. Explain why curvature and torsion are not multiplied by $2t$.

Solution. By the chain rule and $\|c'\| = 1$,

$$\dot{\alpha}(t) = 2t T(t^2), \quad \|\dot{\alpha}(t)\| = 2t, \quad \text{Length}(\alpha) = \int_1^2 2t \, dt = 3.$$

Here $s = t^2$ (up to an additive constant), so $d/ds = (2t)^{-1}d/dt$. The unit tangent of α is $T_\alpha(t) = T(t^2)$. Therefore

$$\frac{dT_\alpha}{ds} = \frac{1}{2t} \frac{d}{dt} T(t^2) = T'(t^2), \quad \kappa_\alpha = \frac{4}{5}.$$

Similarly, $B_\alpha(t) = B(t^2)$ and

$$\frac{dB_\alpha}{ds} = B'(t^2) = -\frac{3}{5}N(t^2), \quad \tau_\alpha = \frac{3}{5}.$$

The extra factor from the chain rule cancels when we differentiate with respect to arclength. Speed depends on the parameter; these geometric quantities do not.

4. A curvature bound for spherical curves.

Suppose $\gamma(s)$ is unit speed and lies on a sphere of radius $R > 0$ centered at m . Prove $\kappa(s) \geq 1/R$. Can the curve of Problem 1 lie on a sphere of radius 1?

Solution. Differentiate $\langle \gamma - m, \gamma - m \rangle = R^2$ twice:

$$\langle \gamma', \gamma - m \rangle = 0, \quad \langle \gamma'', \gamma - m \rangle + \langle \gamma', \gamma' \rangle = 0.$$

Unit speed gives $\langle \gamma', \gamma' \rangle = 1$. Hence the Cauchy–Schwarz inequality yields

$$1 = |\langle \gamma'', \gamma - m \rangle| \leq \|\gamma''\| \|\gamma - m\| = \kappa R.$$

Thus $\kappa \geq 1/R$. Problem 1 has $\kappa = 4/5 < 1$, so it cannot lie on a sphere of radius 1. Notice that unit speed, not the radius of the sphere, is what makes $\langle \gamma', \gamma' \rangle = 1$.

5. Reconstructing a plane curve.

Find a unit-speed plane curve with $c(0) = 0$, $T(0) = (1, 0)$, and signed curvature $k(s) = 2s$. Definite integrals are acceptable. Compute its signed total curvature and total ordinary curvature on $[-1, 1]$. Is it a Frenet curve on the whole interval?

Use $J(x, y) = (-y, x)$ and $T' = kJT$; if $T = (\cos \theta, \sin \theta)$, then $\theta' = k$.

Solution. Take $\theta(0) = 0$. Integration gives

$$\theta(s) = \int_0^s 2r \, dr = s^2, \quad T(s) = (\cos(s^2), \sin(s^2)).$$

Integrating the tangent once more gives the required curve:

$$c(s) = \left(\int_0^s \cos(r^2) \, dr, \int_0^s \sin(r^2) \, dr \right).$$

Indeed, $c' = T$ has norm 1, and $T' = 2sJT$, verifying both unit speed and the prescribed signed curvature. Moreover,

$$\int_{-1}^1 k(s) \, ds = 0, \quad \int_{-1}^1 \kappa(s) \, ds = \int_{-1}^1 2|s| \, ds = 2.$$

It is regular everywhere but not Frenet at $s = 0$, because $\kappa(0) = 0$ and $N = T'/\kappa$ is undefined there.

6. Reversing a plane curve.

Let $c(s)$ be unit speed on a symmetric interval and set $\bar{c}(s) = c(-s)$. With the same fixed rotation J , prove

$$\bar{k}(s) = -k(-s), \quad \bar{\kappa}(s) = \kappa(-s).$$

Solution. The reversed curve is still unit speed, and its unit tangent is $\bar{T}(s) = -T(-s)$. Differentiating carefully,

$$\bar{T}'(s) = T'(-s) = k(-s)JT(-s) = -k(-s)J\bar{T}(s).$$

Comparing this with $\bar{T}' = \bar{k}J\bar{T}$ gives $\bar{k}(s) = -k(-s)$. Taking absolute values gives $\bar{\kappa}(s) = |\bar{k}(s)| = |k(-s)| = \kappa(-s)$. Signed curvature detects the direction of traversal; ordinary curvature does not.

7. Tangent spaces to a level surface.

Let $S = \{x^2 + y^2 - z^2 = 1\}$ and $p = (\sqrt{2}, 0, 1)$. Find $T_p S$, a unit normal, and the affine tangent plane. Briefly justify the formula $T_p S = \ker dF_p$ for $F(x, y, z) = x^2 + y^2 - z^2 - 1$.

Solution. We have $F(p) = 0$ and $\nabla F(p) = 2(\sqrt{2}, 0, -1) \neq 0$, so the level surface is regular near p . Thus

$$T_p S = \{(a, b, c) \in \mathbb{R}^3 : \sqrt{2}a - c = 0\}, \quad \nu(p) = \frac{1}{\sqrt{3}}(\sqrt{2}, 0, -1).$$

The affine tangent plane is $p + T_p S$, with equation

$$\sqrt{2}(x - \sqrt{2}) - (z - 1) = 0, \quad \text{or equivalently} \quad \sqrt{2}x - z = 1.$$

For the justification, represent a tangent vector as $v = \dot{\gamma}(0)$, where $\gamma(0) = p$ and $\gamma(t) \in S$. Since $F(\gamma(t)) = 0$, the chain rule gives

$$dF_p(v) = \left. \frac{d}{dt} F(\gamma(t)) \right|_{t=0} = 0.$$

Hence $T_p S \subseteq \ker dF_p$. The regular level-set theorem gives $\dim T_p S = 2$, and $dF_p \neq 0$ gives $\dim \ker dF_p = 2$. The inclusion is therefore equality.

8. A tangent vector need not give a line in the surface.

For the saddle $S = \{z = xy\}$ at the origin, show that $X = (1, 0, 0)$ and $Y = (1, 1, 0)$ are tangent vectors. Which of the lines tX and tY lies in S ? Find a curve in S whose velocity at 0 is Y .

Solution. For $F(x, y, z) = z - xy$, $\nabla F(0, 0, 0) = (0, 0, 1)$. Consequently,

$$T_0 S = \{(a, b, 0) : a, b \in \mathbb{R}\},$$

which contains both X and Y . The line $tX = (t, 0, 0)$ lies in S since $0 = t \cdot 0$. For $tY = (t, t, 0)$, the surface equation requires $0 = t^2$, so this line meets S only at the origin.

Nevertheless, $\gamma(t) = (t, t, t^2)$ lies in S and $\dot{\gamma}(0) = (1, 1, 0) = Y$. Tangency prescribes a curve's first-order motion, not its entire straight-line continuation.

9. Lengths and angles from the first fundamental form.

For $f(u, v) = (u + v, v, 0)$, find the metric matrix g , the length of $X = f_u - f_v$, and the angle between f_u and f_v . Recall that $g_{ij} = \langle f_i, f_j \rangle$ and a tangent vector with coordinate column a has squared length $a^T g a$.

Solution. Here $f_u = (1, 0, 0)$, $f_v = (1, 1, 0)$, and

$$g = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad \|X\|^2 = \begin{pmatrix} 1 & -1 \end{pmatrix} g \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1.$$

Thus $\|X\| = 1$. The angle θ satisfies

$$\cos \theta = \frac{\langle f_u, f_v \rangle}{\|f_u\| \|f_v\|} = \frac{1}{\sqrt{2}}, \quad \theta = \frac{\pi}{4}.$$

The coordinate basis is not orthonormal: the coordinate pair $(1, -1)$ does not have Euclidean length $\sqrt{2}$ as a tangent vector.

10. The shape operator in a nonunit basis.

At $(u, v) = (0, 0)$, consider

$$f(u, v) = \left(u, 2v, \frac{1}{2}(u + 2v)^2 \right)$$

with the upward normal. Find g , h , and $[L]$, then the principal curvatures and corresponding unit directions in $\mathbb{R}^3$. Use $h_{ij} = \langle f_{ij}, \nu \rangle$ and $[L] = g^{-1}h$.

Solution. At the origin, $f_u = (1, 0, 0)$, $f_v = (0, 2, 0)$, and $\nu = (0, 0, 1)$. The second derivatives are $f_{uu} = (0, 0, 1)$, $f_{uv} = (0, 0, 2)$, and $f_{vv} = (0, 0, 4)$. Therefore

$$g = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}, \quad [L] = \begin{pmatrix} 1 & 2 \\ 1/2 & 1 \end{pmatrix}.$$

Its characteristic polynomial is $(1 - \lambda)^2 - 1$, so the principal curvatures are 2 and 0. Corresponding coordinate eigenvectors are $(2, 1)$ and $(2, -1)$. Convert to tangent vectors before normalizing:

$$2f_u + f_v = (2, 2, 0), \quad 2f_u - f_v = (2, -2, 0).$$

The unit principal directions are $(1, 1, 0)/\sqrt{2}$ and $(1, -1, 0)/\sqrt{2}$, respectively. Diagonalize $g^{-1}h$, not h .

11. Why the shape operator is self-adjoint.

For a smooth regular patch f with unit normal ν , use

$$L(f_i) = -\nu_i, \quad I(X, Y) = \langle X, Y \rangle$$

to prove $I(LX, Y) = I(X, LY)$. Why can the matrix in Problem 10 fail to be symmetric?

Solution. Differentiate $\langle \nu, f_j \rangle = 0$ with respect to u_i :

$$\langle \nu_i, f_j \rangle + \langle \nu, f_{ij} \rangle = 0.$$

Consequently,

$$I(Lf_i, f_j) = -\langle \nu_i, f_j \rangle = \langle \nu, f_{ij} \rangle = \langle \nu, f_{ji} \rangle = I(f_i, Lf_j).$$

By bilinearity, this holds for all tangent vectors X, Y . Equivalently, the second fundamental form $II(X, Y) = I(LX, Y)$ is symmetric.

In coordinates, self-adjointness means

$$[L]^T g = g[L],$$

not necessarily $[L]^T = [L]$. The latter is automatic in an orthonormal basis. Problem 10 uses a nonunit coordinate basis.

12. Reading principal curvatures from the normal.

Consider the cone patch

$$f(u, v) = (v \cos u, v \sin u, 2v), \quad v > 0, \quad \nu = \frac{1}{\sqrt{5}}(2 \cos u, 2 \sin u, -1).$$

Find g and the principal curvatures using $L(f_u) = -\nu_u$ and $L(f_v) = -\nu_v$. Explain why the patch is regular.

Solution. We have

$$f_u = (-v \sin u, v \cos u, 0), \quad f_v = (\cos u, \sin u, 2), \quad g = \begin{pmatrix} v^2 & 0 \\ 0 & 5 \end{pmatrix}.$$

Since $\det g = 5v^2 > 0$, the tangent vectors are linearly independent. Also,

$$\nu_u = \frac{2}{\sqrt{5}}(-\sin u, \cos u, 0) = \frac{2}{\sqrt{5}v}f_u, \quad \nu_v = 0.$$

Thus f_u, f_v are principal directions, with principal curvatures $-2/(\sqrt{5}v)$ and 0, respectively. No second-derivative calculation is needed.

13. Area and a change of orientation.

For the cone patch of Problem 12, find the area of the portion $0 \leq u \leq \pi$, $1 \leq v \leq 3$, using $dA = \sqrt{\det g} \, du \, dv$. If we replace ν by $-\nu$, what happens to g , area, II , L , and the principal curvatures?

Solution. The parametrization is one-to-one on this rectangle: the height determines v , and the first two coordinates determine $u \in [0, \pi]$. Thus the integral gives its surface area without multiplicity. Since $v > 0$,

$$dA = \sqrt{5v^2} \, du \, dv = \sqrt{5}v \, du \, dv,$$

and hence

$$\text{Area} = \int_0^\pi \int_1^3 \sqrt{5}v \, dv \, du = \pi\sqrt{5} \left[\frac{v^2}{2} \right]_1^3 = 4\pi\sqrt{5}.$$

The metric depends only on f_u, f_v , so neither g nor area changes when the normal is reversed. On the other hand,

$$\tilde{h}_{ij} = \langle f_{ij}, -\nu \rangle = -h_{ij}, \quad \tilde{II} = -II, \quad \tilde{L} = -L.$$

The principal directions remain the same and the corresponding principal curvatures change sign: they become $2/(\sqrt{5}v)$ and 0.

14. When does a surface patch lie in a plane?

Let $f : U \rightarrow \mathbb{R}^3$ be a regular patch on a connected open set with a smooth unit normal. Prove that $L \equiv 0$ implies $f(U)$ lies in an affine plane. Is one zero principal curvature at every point enough?

Solution. The equalities $\nu_u = -L(f_u)$ and $\nu_v = -L(f_v)$ show that $D\nu = 0$. Thus ν is locally constant, and connectedness makes it a single constant unit vector n on U . The scalar function $q(u, v) = \langle f(u, v), n \rangle$ then satisfies

$$q_u = \langle f_u, n \rangle = 0, \quad q_v = \langle f_v, n \rangle = 0.$$

Again by connectedness, $q = a$ is constant. Hence

$$f(U) \subseteq \{x \in \mathbb{R}^3 : \langle x, n \rangle = a\},$$

an affine plane.

One zero principal curvature is not enough. The cone of Problem 12 has one zero eigenvalue everywhere, but the other is nonzero. A patch in a plane has constant unit normal locally and therefore $L = 0$; this cone cannot be such a patch.