

Practice Midterm 1: Solutions

MA 563 — Differential Geometry

The exam allows 75 minutes and asks students to choose four of five 20-point problems, for a total of 80 points. Solutions to all five problems are provided here. Primes denote arclength derivatives; dots denote derivatives in an arbitrary parameter. Questions are reproduced verbatim in black; solutions are in muted violet.

Problem 1.

Consider the curve

$$c(t) = (2 \cos t, 2 \sin t, t), \quad t \in \mathbb{R}.$$

(a) [5 points] Compute the speed, the arclength function with $s(0) = 0$, and the length of the segment $0 \leq t \leq 2\pi$.

Solution. For $c(t) = (2 \cos t, 2 \sin t, t)$, we have

$$\dot{c}(t) = (-2 \sin t, 2 \cos t, 1), \quad \ddot{c}(t) = (-2 \cos t, -2 \sin t, 0), \quad \ddot{\ddot{c}}(t) = (2 \sin t, -2 \cos t, 0).$$

The speed is constant:

$$\|\dot{c}(t)\| = \sqrt{4 \sin^2 t + 4 \cos^2 t + 1} = \sqrt{5}.$$

Consequently the arclength coordinate satisfying $s(0) = 0$ is

$$s(t) = \int_0^t \|\dot{c}(u)\| du = \sqrt{5} t, \quad L(c|_{[0, 2\pi]}) = 2\pi\sqrt{5}.$$

(b) [9 points] Compute the curvature κ and torsion τ . You may use the standard formulas for an arbitrary regular parameter.

Solution. The cross product and its squared norm are

$$\dot{c} \times \ddot{c} = (2 \sin t, -2 \cos t, 4), \quad \|\dot{c} \times \ddot{c}\|^2 = 20.$$

Thus

$$\kappa = \frac{\|\dot{c} \times \ddot{c}\|}{\|\dot{c}\|^3} = \frac{2\sqrt{5}}{5\sqrt{5}} = \frac{2}{5}.$$

The scalar triple product is

$$\det(\dot{c}, \ddot{c}, \ddot{\ddot{c}}) = \langle (2 \sin t, -2 \cos t, 4), (2 \sin t, -2 \cos t, 0) \rangle = 4.$$

Therefore

$$\tau = \frac{\det(\dot{c}, \ddot{c}, \ddot{\ddot{c}})}{\|\dot{c} \times \ddot{c}\|^2} = \frac{1}{5}.$$

(c) [6 points] Let $\tilde{c} : \mathbb{R} \rightarrow \mathbb{R}^3$ be a unit-speed Frenet curve with the same curvature and torsion functions as the arclength parametrization of c . Explain precisely what the fundamental theorem of curves implies about the relation between these two curves.

Solution. The unit-speed version of the given curve is

$$\hat{c}(s) = c(s/\sqrt{5}) = (2 \cos(s/\sqrt{5}), 2 \sin(s/\sqrt{5}), s/\sqrt{5}).$$

By the fundamental theorem of curves, there exist a constant matrix $Q \in \text{SO}(3)$ and a constant vector $a \in \mathbb{R}^3$ such that

$$\tilde{c}(s) = Q\hat{c}(s) + a \quad \text{for every } s \in \mathbb{R}.$$

Here $Q^T Q = I$ and $\det Q = 1$. The curves agree up to a rotation and translation, with the same arclength parameter. Matching their initial points and positively oriented Frenet frames determines this motion. With the same initial data they coincide. A reflection would reverse the sign of the nonzero torsion and is not allowed.

Problem 2.

Let $c : I \rightarrow \mathbb{R}^3$ be a smooth unit-speed Frenet curve on an interval, with $\kappa > 0$ and $\tau \neq 0$ everywhere. Write $\rho = 1/\kappa$. You may use the Frenet equations

$$T' = \kappa N, \quad N' = -\kappa T + \tau B, \quad B' = -\tau N.$$

(a) [10 points] Suppose c lies on a sphere with fixed center m . By differentiating $\|c - m\|^2$, show that

$$m = c + \rho N + \frac{\rho'}{\tau} B.$$

Solution. Let c be unit speed, let $\kappa > 0$ and $\tau \neq 0$, and write $\rho = 1/\kappa$. Its Frenet frame satisfies

$$T' = \kappa N, \quad N' = -\kappa T + \tau B, \quad B' = -\tau N.$$

Suppose $\|c(s) - m\|^2 = R^2$, where m and R are constant. Set $q = m - c$, so $q' = -T$. Differentiating the squared distance gives

$$0 = \frac{d}{ds} \langle q, q \rangle = -2 \langle q, T \rangle.$$

Thus q has no component in the tangent direction. Since (T, N, B) is an orthonormal basis, write

$$q = \alpha N + \beta B.$$

Differentiate $\langle q, T \rangle = 0$:

$$0 = \langle -T, T \rangle + \langle q, \kappa N \rangle = -1 + \kappa \alpha.$$

Hence $\alpha = 1/\kappa = \rho$. To find the binormal component, differentiate $\langle q, N \rangle = \rho$:

$$\begin{aligned} \rho' &= \langle q', N \rangle + \langle q, N' \rangle \\ &= \langle -T, N \rangle + \langle q, -\kappa T + \tau B \rangle \\ &= \tau \langle q, B \rangle = \tau \beta. \end{aligned}$$

Since $\tau \neq 0$, we obtain $\beta = \rho'/\tau$, and therefore

$$m = c + \rho N + \frac{\rho'}{\tau} B.$$

Each component of the center vector has been forced by the constant distance condition: its tangent component is zero, its normal component is ρ , and its binormal component is ρ'/τ .

(b) [10 points] Prove that c lies on a sphere if and only if

$$\left(\frac{\rho'}{\tau}\right)' + \rho\tau = 0.$$

In the converse direction, explain why both the center and the distance from c to that center are constant. Do not quote the spherical-curve criterion.

Solution. Define the candidate center along any such curve by

$$m(s) = c(s) + \rho(s)N(s) + q(s)B(s), \quad q = \frac{\rho'}{\tau}.$$

Using the product rule and the Frenet equations,

$$\begin{aligned} m' &= T + \rho'N + \rho(-\kappa T + \tau B) + q'B - q\tau N \\ &= (1 - \rho\kappa)T + (\rho' - q\tau)N + (\rho\tau + q')B \\ &= \left[\rho\tau + \left(\frac{\rho'}{\tau}\right)'\right]B. \end{aligned}$$

The tangent and normal components vanish because $\rho\kappa = 1$ and $q\tau = \rho'$.

Necessity. If c lies on a sphere, part (a) identifies its fixed center with this $m(s)$. Hence $m' = 0$. Since B is a unit vector,

$$\left(\frac{\rho'}{\tau}\right)' + \rho\tau = 0.$$

Sufficiency. Conversely, suppose this equation holds. The calculation above gives $m' = 0$, so $m(s)$ is a fixed point on the interval I . Furthermore,

$$m - c = \rho N + qB \perp T.$$

It follows that

$$\frac{d}{ds}\|m - c\|^2 = -2\langle m - c, T \rangle = 0.$$

Thus $\|m - c\| = R$ is constant. It is positive because

$$R^2 = \rho^2 + q^2 \geq \rho^2 > 0.$$

Therefore the entire curve lies on the sphere with center m and radius R . This proves both directions without appealing to the spherical-curve criterion as a previously established theorem.

Problem 3.

Let

$$f(u, v) = (u, v, uv), \quad (u, v) \in \mathbb{R}^2,$$

and let $M = f(\mathbb{R}^2)$.

(a) [6 points] Show that f is an embedding, and find the affine tangent plane to M at $p = f(1, 1) = (1, 1, 1)$.

Solution. Consider $f(u, v) = (u, v, uv)$ and $M = f(\mathbb{R}^2)$.

We have

$$f_u = (1, 0, v), \quad f_v = (0, 1, u), \quad f_u \times f_v = (-v, -u, 1).$$

The cross product never vanishes, so f is an immersion. It is injective because its first two coordinates recover (u, v) . Its inverse on M is the restriction of the continuous projection $(x, y, z) \mapsto (x, y)$. Thus f is a homeomorphism onto its image and an immersion, hence an embedding.

At $(u, v) = (1, 1)$, the tangent space is spanned by $(1, 0, 1)$ and $(0, 1, 1)$, with normal vector $(-1, -1, 1)$. The affine tangent plane through $p = (1, 1, 1)$ is therefore

$$-(x - 1) - (y - 1) + (z - 1) = 0, \quad \boxed{z = x + y - 1.}$$

(b) [6 points] Compute E, F, G and show that the metric is positive definite.

Solution. Taking the inner products of the coordinate tangent vectors gives

$$\boxed{E = 1 + v^2, \quad F = uv, \quad G = 1 + u^2.}$$

The metric matrix and quadratic form are

$$g(u, v) = \begin{pmatrix} 1 + v^2 & uv \\ uv & 1 + u^2 \end{pmatrix},$$

$$ds^2 = (1 + v^2) du^2 + 2uv du dv + (1 + u^2) dv^2.$$

We have $E > 0$ and

$$EG - F^2 = (1 + v^2)(1 + u^2) - u^2v^2 = 1 + u^2 + v^2 > 0.$$

These are the principal-minor conditions for a symmetric 2×2 matrix to be positive definite. Alternatively, for every $(a, b) \neq (0, 0)$,

$$(a \ b) g \begin{pmatrix} a \\ b \end{pmatrix} = a^2 + b^2 + (va + ub)^2 > 0.$$

(c) [3 points] Compute the angle between f_u and f_v at $(1, 1)$.

Solution. At $(1, 1)$, we have $E = G = 2$ and $F = 1$. If $\alpha \in [0, \pi]$ is the angle between f_u and f_v , then

$$\cos \alpha = \frac{\langle f_u, f_v \rangle}{\|f_u\| \|f_v\|} = \frac{F}{\sqrt{EG}} = \frac{1}{2}.$$

Consequently,

$$\boxed{\alpha = \frac{\pi}{3}.$$

(d) [5 points] Set $\phi(r, s) = (r + s, r - s)$ and $\tilde{f} = f \circ \phi$. Use the coordinate-change formula to compute the matrix of the first fundamental form of $\tilde{f}$ at $(r, s) = (1, 0)$. Are its coordinate tangent vectors orthogonal there?

Solution. For $\phi(r, s) = (r + s, r - s)$, the Jacobian is constant:

$$A = D\phi = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \det A = -2.$$

It is invertible; in fact, $\phi^{-1}(u, v) = ((u + v)/2, (u - v)/2)$, so this is a valid global change of coordinates.

The point $(r, s) = (1, 0)$ maps to $(u, v) = (1, 1)$. The coordinate-change formula must therefore use $g(1, 1)$:

$$\begin{aligned} \tilde{g}(1, 0) &= A^T g(\phi(1, 0)) A \\ &= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 3 & -1 \end{pmatrix} = \boxed{\begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix}}. \end{aligned}$$

The new coordinate tangent vectors are orthogonal because the off-diagonal entry is zero. As a direct check,

$$\tilde{f}(r, s) = (r + s, r - s, r^2 - s^2),$$

so at $(1, 0)$,

$$\tilde{f}_r = (1, 1, 2), \quad \tilde{f}_s = (1, -1, 0).$$

Their squared lengths are 6 and 2, and their inner product is zero, agreeing with the matrix calculation.

Problem 4.

Consider the helicoid

$$f(u, v) = (u \cos v, u \sin v, v), \quad (u, v) \in \mathbb{R}^2.$$

Use the unit normal $\nu = (f_u \times f_v) / \|f_u \times f_v\|$.

(a) [7 points] Show that f is an immersion. Compute E, F, G and ν .

Solution. For the helicoid $f(u, v) = (u \cos v, u \sin v, v)$, the normal is chosen by the ordered cross product $f_u \times f_v$.

The first derivatives are

$$f_u = (\cos v, \sin v, 0), \quad f_v = (-u \sin v, u \cos v, 1).$$

Their cross product is

$$f_u \times f_v = (\sin v, -\cos v, u), \quad \|f_u \times f_v\| = \sqrt{1 + u^2} > 0.$$

Thus f is an immersion everywhere. The required unit normal is

$$\nu = \frac{(\sin v, -\cos v, u)}{\sqrt{1 + u^2}}.$$

The first fundamental form coefficients are

$$E = \cos^2 v + \sin^2 v = 1,$$

$$F = -u \cos v \sin v + u \sin v \cos v = 0,$$

$$G = u^2 \sin^2 v + u^2 \cos^2 v + 1 = 1 + u^2.$$

In particular,

$$ds^2 = du^2 + (1 + u^2) dv^2.$$

(b) [5 points] Compute the coefficients of the second fundamental form:

$$e = \langle f_{uu}, \nu \rangle, \quad b = \langle f_{uv}, \nu \rangle, \quad g = \langle f_{vv}, \nu \rangle.$$

Solution. The second derivatives are

$$f_{uu} = (0, 0, 0), \quad f_{uv} = (-\sin v, \cos v, 0), \quad f_{vv} = (-u \cos v, -u \sin v, 0).$$

Taking their inner products with the specified ν ,

$$e = 0, \quad b = \frac{-\sin^2 v - \cos^2 v}{\sqrt{1 + u^2}} = -\frac{1}{\sqrt{1 + u^2}},$$

$$g = \frac{-u \cos v \sin v + u \sin v \cos v}{\sqrt{1+u^2}} = 0.$$

Thus the second fundamental form is

$$\Pi = -\frac{2}{\sqrt{1+u^2}} du dv.$$

The negative sign of b is determined by the chosen normal orientation.

(c) [5 points] Compute the Gaussian curvature and mean curvature at every point. You may use

$$K = \frac{eg - b^2}{EG - F^2}, \quad H = \frac{eG - 2bF + gE}{2(EG - F^2)}.$$

Solution. Using $E = 1$, $F = 0$, $G = 1 + u^2$, we have

$$EG - F^2 = 1 + u^2.$$

Since $e = g = 0$ and $b = -1/\sqrt{1+u^2}$,

$$eg - b^2 = -\frac{1}{1+u^2}.$$

Consequently,

$$K(u, v) = \frac{eg - b^2}{EG - F^2} = -\frac{1}{(1+u^2)^2}.$$

For mean curvature the numerator is zero:

$$eG - 2bF + gE = 0,$$

and therefore

$$H(u, v) = \frac{eG - 2bF + gE}{2(EG - F^2)} = 0.$$

In particular, the helicoid has negative Gaussian curvature at every point, even though its mean curvature vanishes everywhere.

(d) [3 points] Explain why the helicoid is both a ruled surface and a minimal surface. Identify which coordinate curves are the straight rulings.

Solution. Fix $v = v_0$. Then

$$f(u, v_0) = (0, 0, v_0) + u(\cos v_0, \sin v_0, 0).$$

As u varies, this is a straight line at height v_0 . These lines cover the helicoid, so it is a ruled surface. The straight rulings are the u -coordinate curves, with v held fixed.

A regular surface is minimal precisely when its mean curvature vanishes identically. Since $H \equiv 0$, the helicoid is minimal. This is a statement about mean curvature (equivalently, stationary area under compactly supported normal variations); it does not say that the Gaussian curvature is zero.

Problem 5.

At a point p of an oriented surface, let e_1, e_2 be an orthonormal basis of principal directions, with principal curvatures k_1, k_2 . The normal curvature in a unit direction X is $k_n(X) = \langle SX, X \rangle$. A nonzero direction with zero normal curvature is called *asymptotic*.

(a) [5 points] For $X = \cos \theta e_1 + \sin \theta e_2$, derive

$$k_n(X) = k_1 \cos^2 \theta + k_2 \sin^2 \theta.$$

Solution. At the point in question, e_1, e_2 form an orthonormal basis with $Se_1 = k_1 e_1$ and $Se_2 = k_2 e_2$.

For the unit vector $X = \cos \theta e_1 + \sin \theta e_2$,

$$SX = k_1 \cos \theta e_1 + k_2 \sin \theta e_2.$$

By orthonormality, the cross terms in $\langle SX, X \rangle$ vanish, and hence

$$\boxed{k_n(X) = \langle SX, X \rangle = k_1 \cos^2 \theta + k_2 \sin^2 \theta.}$$

This derives Euler's formula directly from the eigenvector equations.

(b) [10 points] Assume $K(p) < 0$, and label the principal curvatures so that $k_1 > 0 > k_2$. Show that there are exactly two distinct unoriented asymptotic tangent lines at p , and prove that they are perpendicular if and only if $H(p) = 0$.

Solution. Assume $K = k_1 k_2 < 0$, and label the principal curvatures with $k_1 > 0 > k_2$. A nonzero tangent vector $V = xe_1 + ye_2$ determines an asymptotic line exactly when

$$\langle SV, V \rangle = k_1 x^2 + k_2 y^2 = 0.$$

Normalization does not affect this zero condition. Neither coordinate can vanish: if $x = 0$ or $y = 0$, the equation would force $V = 0$. Dividing by x^2 gives

$$\left(\frac{y}{x}\right)^2 = -\frac{k_1}{k_2}.$$

Let $a = \sqrt{-k_1/k_2} > 0$. There are exactly two distinct unoriented lines, spanned by

$$V_+ = e_1 + ae_2, \quad V_- = e_1 - ae_2.$$

Their inner product is

$$\langle V_+, V_- \rangle = 1 - a^2 = 1 + \frac{k_1}{k_2} = \frac{k_1 + k_2}{k_2}.$$

Since $k_2 \neq 0$, these lines are perpendicular if and only if $k_1 + k_2 = 0$, which is equivalent to

$$\boxed{H = \frac{k_1 + k_2}{2} = 0.}$$

This is a pointwise statement. If $K < 0$ everywhere, the asymptotic lines are perpendicular everywhere if and only if the surface is minimal.

(c) [5 points] After reversing the unit normal, how do the principal curvatures, K , H , and the asymptotic tangent lines change? Justify your answers.

Solution. Replacing ν by $-\nu$ changes the shape operator to $\tilde{S} = -d(-\nu) = -S$. Relative to the same principal directions, the principal curvatures become $-k_1, -k_2$. Therefore

$$\boxed{\tilde{K} = K, \quad \tilde{H} = -H.}$$

If they are relabeled so that the positive curvature is first, their labels exchange. The asymptotic lines themselves do not change, since $\langle \tilde{S}V, V \rangle = -\langle SV, V \rangle$ has exactly the same zeros. In particular, minimality and the angle between the asymptotic lines are independent of normal orientation.