

Practice Midterm 2: Solutions

MA 563 — Differential Geometry

The exam allows 75 minutes: choose four of five 20-point problems, for 80 points total. Full solutions to all five problems follow. Questions are reproduced verbatim in black; solutions are in muted violet. Dots denote arbitrary-parameter derivatives and primes denote arclength derivatives. We use $L = -D\nu \circ (Df)^{-1}$.

Problem 1.

Consider the space curve

$$c(t) = \left(t, t^2, \frac{2}{3}t^3\right), \quad t \in \mathbb{R}.$$

(a) [5 points] Compute the speed and the arclength function with $s(0) = 0$. Find the length of the segment $0 \leq t \leq 1$.

Solution. Differentiate to obtain

$$\dot{c} = (1, 2t, 2t^2), \quad \|\dot{c}\| = \sqrt{1 + 4t^2 + 4t^4} = 1 + 2t^2.$$

The speed is positive, so the curve is regular. Hence

$$s(t) = \int_0^t (1 + 2r^2) dr = t + \frac{2}{3}t^3, \quad \text{Length}(c|_{[0,1]}) = s(1) - s(0) = \frac{5}{3}.$$

(b) [7 points] Compute $\kappa(t)$ and $\tau(t)$. You may use

$$\kappa = \frac{\|\dot{c} \times \ddot{c}\|}{\|\dot{c}\|^3}, \quad \tau = \frac{\det(\dot{c}, \ddot{c}, \ddot{\ddot{c}})}{\|\dot{c} \times \ddot{c}\|^2}.$$

Solution. The next derivatives and the cross product are

$$\ddot{c} = (0, 2, 4t), \quad \ddot{\ddot{c}} = (0, 0, 4), \quad \dot{c} \times \ddot{c} = (4t^2, -4t, 2).$$

Therefore

$$\|\dot{c} \times \ddot{c}\|^2 = 16t^4 + 16t^2 + 4 = 4(1 + 2t^2)^2,$$

and the scalar triple product is

$$\det(\dot{c}, \ddot{c}, \ddot{\ddot{c}}) = \langle (4t^2, -4t, 2), (0, 0, 4) \rangle = 8.$$

Substitution gives

$$\kappa(t) = \frac{2(1 + 2t^2)}{(1 + 2t^2)^3} = \frac{2}{(1 + 2t^2)^2}, \quad \tau(t) = \frac{8}{4(1 + 2t^2)^2} = \frac{2}{(1 + 2t^2)^2}.$$

In particular, $\kappa > 0$ everywhere, so the Frenet frame is defined.

Problem 1 (continued).**(c) [3 points]** Compute the Frenet frame T, N, B at $t = 0$.**Solution.** The unit tangent is

$$T(t) = \frac{(1, 2t, 2t^2)}{1 + 2t^2}, \quad T(0) = (1, 0, 0).$$

We must differentiate with respect to arclength when finding N . Since $ds/dt = 1 + 2t^2$, the quotient rule gives

$$\left. \frac{dT}{dt} \right|_{t=0} = (0, 2, 0), \quad \left. \frac{dT}{ds} \right|_{t=0} = \frac{1}{1 + 2t^2} \left. \frac{dT}{dt} \right|_{t=0} = (0, 2, 0).$$

Using $\kappa(0) = 2$, we obtain

$$N(0) = \frac{1}{\kappa} \left. \frac{dT}{ds} \right|_{t=0} = (0, 1, 0), \quad B(0) = T(0) \times N(0) = (0, 0, 1).$$

Thus

$$\boxed{T(0) = (1, 0, 0), \quad N(0) = (0, 1, 0), \quad B(0) = (0, 0, 1).}$$

(d) [5 points] Let $\gamma(s)$ be any unit-speed Frenet curve on an interval satisfying $\tau(s) = \kappa(s) > 0$. Prove that

$$A(s) = \frac{T(s) + B(s)}{\sqrt{2}}$$

is a constant unit vector, and that T makes an angle $\pi/4$ with this fixed vector. You may use the Frenet equations, but not the slope-line theorem.

Solution. The Frenet equations $T' = \kappa N$ and $B' = -\tau N$ give

$$A' = \frac{T' + B'}{\sqrt{2}} = \frac{\kappa - \tau}{\sqrt{2}} N = 0.$$

Because the parameter domain is an interval, A is constant. The frame is orthonormal, so

$$\|A\|^2 = \frac{\|T\|^2 + 2\langle T, B \rangle + \|B\|^2}{2} = 1, \quad \langle T, A \rangle = \frac{1}{\sqrt{2}}.$$

Both vectors are unit vectors; hence the angle is $\arccos(1/\sqrt{2}) = \pi/4$ at every point. For the particular curve in this problem, the fixed direction is $A = (1, 0, 1)/\sqrt{2}$, as follows from part (c).

Problem 2.

Consider the plane curve

$$c(t) = (e^t \cos t, e^t \sin t), \quad t \in \mathbb{R}.$$

Use $J(x, y) = (-y, x)$, so that signed curvature is defined by $dT/ds = kJT$.

(a) [6 points] Compute the speed and $s(t)$, with $s(0) = 0$. Find a unit-speed parametrization $\widehat{c}(s)$, including its parameter interval.

Solution. We have

$$\dot{c}(t) = e^t(\cos t - \sin t, \sin t + \cos t), \quad \|\dot{c}(t)\| = \sqrt{2}e^t.$$

Consequently

$$s(t) = \int_0^t \sqrt{2} e^r dr = \sqrt{2}(e^t - 1).$$

This strictly increasing function maps $\mathbb{R}$ onto $(-\sqrt{2}, \infty)$, and its inverse is

$$t(s) = \log \left(1 + \frac{s}{\sqrt{2}} \right).$$

Put $q(s) = 1 + s/\sqrt{2} > 0$. An explicit arclength parametrization is

$$\widehat{c}(s) = q(s)(\cos(\log q(s)), \sin(\log q(s))), \quad -\sqrt{2} < s < \infty.$$

Indeed, $dt/ds = 1/(\sqrt{2}e^{t(s)})$, so the chain rule yields $\|\widehat{c}'(s)\| = \|\dot{c}(t(s))\| dt/ds = 1$. The finite left endpoint reflects the finite length toward the origin; the origin itself is not on the parametrized curve.

(b) [5 points] Compute the signed curvature $k(t)$ and the nonnegative curvature $\kappa(t)$. You may use

$$k(t) = \frac{\det(\dot{c}(t), \ddot{c}(t))}{\|\dot{c}(t)\|^3}.$$

Solution. Differentiate again:

$$\ddot{c}(t) = 2e^t(-\sin t, \cos t).$$

Then

$$\begin{aligned} \det(\dot{c}, \ddot{c}) &= 2e^{2t}[(\cos t - \sin t) \cos t + (\sin t + \cos t) \sin t] \\ &= 2e^{2t}. \end{aligned}$$

Thus

$$k(t) = \frac{2e^{2t}}{(\sqrt{2}e^t)^3} = \frac{1}{\sqrt{2}e^t}, \quad \kappa(t) = |k(t)| = \frac{1}{\sqrt{2}e^t}.$$

Problem 2 (continued).

(c) [4 points] Compute the signed total curvature of the segment $0 \leq t \leq \pi$, and the length of that segment.

Solution. Use the arclength element $ds = \|\dot{c}(t)\| dt = \sqrt{2}e^t dt$. The signed total curvature is

$$\int_{c|_{[0,\pi]}} k ds = \int_0^\pi \frac{1}{\sqrt{2}e^t} \sqrt{2}e^t dt = \int_0^\pi 1 dt = \pi.$$

The length is

$$\text{Length}(c|_{[0,\pi]}) = \int_0^\pi \sqrt{2}e^t dt = \sqrt{2}(e^\pi - 1).$$

The speed factor in the first integral is essential: total curvature integrates curvature with respect to arclength, not the arbitrary parameter t .

(d) [5 points] Let $\gamma(s)$ be any unit-speed plane curve on an interval with constant signed curvature $k_0 \neq 0$. Prove that its image lies on a circle of radius $1/|k_0|$. *Hint:* Differentiate $\gamma + k_0^{-1}JT$.

Solution. Since $\gamma' = T$ and $T' = k_0 JT$, define the candidate center

$$m(s) = \gamma(s) + \frac{1}{k_0} JT(s).$$

The map J is constant and satisfies $J^2 = -\text{id}$. Differentiating, using that k_0 is constant, gives

$$\begin{aligned} m' &= T + \frac{1}{k_0} JT' = T + \frac{1}{k_0} J(k_0 JT) \\ &= T + J^2 T = T - T = 0. \end{aligned}$$

Thus m is a fixed point on the interval. Also, J preserves length and $\|T\| = 1$, so

$$\|\gamma(s) - m\| = \left\| -\frac{1}{k_0} JT(s) \right\| = \frac{1}{|k_0|}.$$

The image therefore lies on the circle centered at m with radius $1/|k_0|$. It may trace only an arc; no closedness is assumed.

Problem 3.

Let

$$f(u, v) = \left(u, v, \frac{u^2 + v^2}{2} \right), \quad M = f(\mathbb{R}^2).$$

(a) [5 points] Compute f_u, f_v and the upward unit normal. Find the affine tangent plane at $p = f(1, 0)$.

Solution. The coordinate tangent vectors and their cross product are

$$f_u = (1, 0, u), \quad f_v = (0, 1, v), \quad f_u \times f_v = (-u, -v, 1).$$

The positive third component selects the upward orientation. Hence

$$\nu(u, v) = \frac{(-u, -v, 1)}{\sqrt{1 + u^2 + v^2}}.$$

At $p = (1, 0, 1/2)$, the tangent space is spanned by $(1, 0, 1)$ and $(0, 1, 0)$; a normal is $(-1, 0, 1)$. The affine tangent plane is therefore

$$\langle (-1, 0, 1), (x, y, z) - (1, 0, 1/2) \rangle = 0,$$

or, equivalently,

$$z = x - \frac{1}{2}.$$

(b) [5 points] Compute the matrix $g = (g_{ij})$ of the first fundamental form for arbitrary (u, v) . At $(1, 0)$, compute the length of $X = f_u + 2f_v$ and the angle between f_u and f_v .

Solution. By taking inner products of the coordinate tangent vectors,

$$g_{11} = 1 + u^2, \quad g_{12} = g_{21} = uv, \quad g_{22} = 1 + v^2.$$

Thus

$$g(u, v) = \begin{pmatrix} 1 + u^2 & uv \\ uv & 1 + v^2 \end{pmatrix}, \quad g(1, 0) = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

The coordinate column of X is $(1, 2)^T$, so

$$\|X\|^2 = \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 6, \quad \|X\| = \sqrt{6}.$$

Equivalently, $X = (1, 2, 1)$ in $\mathbb{R}^3$. Finally, $\langle f_u, f_v \rangle = 0$ at $(1, 0)$, and both vectors are nonzero. Hence their angle is $\pi/2$.

Problem 3 (continued).**(c) [5 points]** Compute the area of the surface region

$$f(\{(u, v) : u^2 + v^2 \leq 1\}).$$

You may use $dA = \sqrt{\det g} du dv$ and polar coordinates.**Solution.** The determinant simplifies to

$$\det g = (1 + u^2)(1 + v^2) - u^2v^2 = 1 + u^2 + v^2.$$

The patch is injective, so the area integral counts each surface point once. Using $u = r \cos \theta$, $v = r \sin \theta$, we obtain

$$\begin{aligned} \text{Area} &= \iint_{u^2+v^2 \leq 1} \sqrt{1 + u^2 + v^2} du dv \\ &= \int_0^{2\pi} \int_0^1 \sqrt{1 + r^2} r dr d\theta \\ &= \frac{2\pi}{3} [(1 + r^2)^{3/2}]_{r=0}^{r=1}. \end{aligned}$$

Therefore

$$\boxed{\text{Area} = \frac{2\pi}{3}(2\sqrt{2} - 1).}$$

The factor r comes from the planar polar-coordinate Jacobian; it is in addition to the surface-area factor $\sqrt{1 + r^2}$.**(d) [5 points]** Prove that $f : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is an embedding: show that it is an immersion and a homeomorphism onto its image.**Solution.** The map f is smooth because its components are polynomials. The calculation in part (a) gives

$$f_u \times f_v = (-u, -v, 1) \neq 0$$

at every point. Thus f_u, f_v are linearly independent and Df has rank two everywhere: f is an immersion.If $f(u, v) = f(\tilde{u}, \tilde{v})$, comparison of the first two coordinates gives $u = \tilde{u}$, $v = \tilde{v}$. Hence f is injective. Its inverse on its image M is

$$f^{-1} : M \longrightarrow \mathbb{R}^2, \quad (x, y, z) \longmapsto (x, y),$$

the restriction of a continuous coordinate projection. Since f is also continuous, it is a homeomorphism onto M , equipped with the subspace topology. Together with immersion, this proves that f is an embedding.

Problem 4.

Consider the cylinder patch

$$f(u, v) = (2 \cos u, 2 \sin u, u + v), \quad (u, v) \in (-\pi, \pi) \times \mathbb{R},$$

with the outward unit normal.

(a) [5 points] Compute f_u, f_v , the unit normal ν , and the matrix of the first fundamental form in the ordered basis (f_u, f_v) .

Solution. Differentiating gives

$$f_u = (-2 \sin u, 2 \cos u, 1), \quad f_v = (0, 0, 1).$$

Their cross product is $(2 \cos u, 2 \sin u, 0)$, of length two, so the outward unit normal is

$$\nu(u, v) = (\cos u, \sin u, 0).$$

The inner products are

$$\langle f_u, f_u \rangle = 5, \quad \langle f_u, f_v \rangle = 1, \quad \langle f_v, f_v \rangle = 1.$$

Therefore

$$g = \begin{pmatrix} 5 & 1 \\ 1 & 1 \end{pmatrix}.$$

The coordinate tangent basis is not orthogonal, because its mixed inner product is one.

(b) [5 points] Compute $L(f_u)$ and $L(f_v)$. Express these vectors in the basis (f_u, f_v) , and write the matrix $[L]$ in that basis.

Solution. We have $\nu_u = (-\sin u, \cos u, 0)$ and $\nu_v = 0$. Using the specified sign convention,

$$\begin{aligned} L(f_u) &= -\nu_u = (\sin u, -\cos u, 0) = -\frac{1}{2}f_u + \frac{1}{2}f_v, \\ L(f_v) &= -\nu_v = 0. \end{aligned}$$

The columns of the matrix are the coordinate columns of these two images, in the stated order. Hence

$$[L] = A = \begin{pmatrix} -\frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{pmatrix}.$$

This matrix is not symmetric in the ordinary transpose sense; its coordinate basis is not orthonormal.

Problem 4 (continued).

(c) [5 points] Find both principal curvatures and a unit principal direction in $\mathbb{R}^3$ for each.

Solution. The characteristic equation is

$$\det(A - \lambda I_2) = (-\tfrac{1}{2} - \lambda)(-\lambda) = \lambda(\lambda + \tfrac{1}{2}) = 0.$$

Thus the principal curvatures are $\boxed{-1/2 \text{ and } 0}$. For $\lambda = -1/2$, an eigenvector in coordinate components is $(1, -1)^\top$. The corresponding tangent vector is

$$f_u - f_v = (-2 \sin u, 2 \cos u, 0),$$

which has length two, not $\sqrt{2}$. For $\lambda = 0$, we may take the coordinate column $(0, 1)^\top$, corresponding to f_v . Unit principal directions in ambient coordinates are therefore

$$\boxed{e_1 = (-\sin u, \cos u, 0), \quad Le_1 = -\tfrac{1}{2}e_1,}$$

$$\boxed{e_2 = (0, 0, 1), \quad Le_2 = 0.}$$

The first direction runs around the cylinder and the second runs parallel to its axis. These unit vectors are perpendicular.

(d) [5 points] You may use that L is self-adjoint for the first fundamental form I . Prove that if $LX = \lambda X$ and $LY = \mu Y$, with $\lambda \neq \mu$, then $I(X, Y) = 0$. Explain why this does not require the matrix in part (b) to be an ordinary symmetric matrix.

Solution. Self-adjointness and bilinearity imply

$$\lambda I(X, Y) = I(LX, Y) = I(X, LY) = \mu I(X, Y).$$

Since $\lambda - \mu \neq 0$, it follows that $I(X, Y) = 0$.

If x, y are coordinate columns in a basis with metric matrix g , then $I(X, Y) = x^\top g y$. Accordingly, self-adjointness of a matrix A means

$$x^\top A^\top g y = x^\top g A y \quad \text{for all } x, y, \quad \text{or} \quad A^\top g = g A.$$

In an orthonormal basis, where $g = I_2$, this reduces to $A^\top = A$. In this example,

$$A^\top g = g A = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix},$$

although $A \neq A^\top$. Orthogonality is measured by the surface metric, not by the ordinary dot product of coordinate columns.

Problem 5.

Let

$$f(u, v) = (u, v, u^2 + uv + v^2), \quad p = f(0, 0),$$

and choose the upward unit normal.

(a) [5 points] Compute the normal field $\nu(u, v)$. Compute $f_u(0, 0)$, $f_v(0, 0)$ and the first fundamental form matrix at $(0, 0)$.

Solution. For convenience, put

$$a = 2u + v, \quad b = u + 2v, \quad W = \sqrt{1 + a^2 + b^2}.$$

Then $f_u = (1, 0, a)$, $f_v = (0, 1, b)$, and $f_u \times f_v = (-a, -b, 1)$. Hence

$$\nu(u, v) = \frac{(-(2u + v), -(u + 2v), 1)}{\sqrt{1 + (2u + v)^2 + (u + 2v)^2}}.$$

At the origin,

$$f_u(0, 0) = (1, 0, 0), \quad f_v(0, 0) = (0, 1, 0), \quad \nu(0, 0) = (0, 0, 1).$$

The tangent basis is orthonormal, so the first fundamental form is

$$g(0, 0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2.$$

(b) [5 points] Differentiate ν at $(0, 0)$ and use $L(f_u) = -\nu_u$, $L(f_v) = -\nu_v$ to find the matrix of L_p in the basis $(f_u(0, 0), f_v(0, 0))$.

Solution. The derivatives of the denominator are

$$W_u = \frac{2a + b}{W}, \quad W_v = \frac{a + 2b}{W}.$$

At $(0, 0)$, we have $a = b = 0$, $W = 1$, and $W_u = W_v = 0$. The quotient rule therefore gives

$$\begin{aligned} \nu_u &= \frac{(-2, -1, 0)}{W} - \frac{W_u}{W^2}(-a, -b, 1), \\ \nu_v &= \frac{(-1, -2, 0)}{W} - \frac{W_v}{W^2}(-a, -b, 1), \end{aligned}$$

so $\nu_u(0, 0) = (-2, -1, 0)$ and $\nu_v(0, 0) = (-1, -2, 0)$. At p , this yields

$$L_p(f_u) = 2f_u + f_v, \quad L_p(f_v) = f_u + 2f_v, \quad [L_p] = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Problem 5 (continued).

(c) [5 points] Find the principal curvatures at p . For each, give a unit principal direction as a vector in $\mathbb{R}^3$.

Solution. The characteristic polynomial is

$$\det([L_p] - \lambda I_2) = (2 - \lambda)^2 - 1 = (\lambda - 3)(\lambda - 1).$$

Thus the principal curvatures are 3 and 1. Their coordinate eigenvectors may be chosen as $(1, 1)^\top$ and $(1, -1)^\top$, respectively. Because f_u, f_v are orthonormal at this point, both corresponding tangent vectors have length $\sqrt{2}$. The requested unit directions are

$$e_1 = \frac{(1, 1, 0)}{\sqrt{2}}, \quad L_p e_1 = 3e_1, \quad e_2 = \frac{(1, -1, 0)}{\sqrt{2}}, \quad L_p e_2 = e_2.$$

These signs use the upward normal and $L = -D\nu \circ (Df)^{-1}$. Reversing the normal would reverse both eigenvalues, but not their eigenlines.

(d) [5 points] Let $\tilde{f} : U \rightarrow M \subset \mathbb{R}^3$ be any regular surface element, with unit normal $\tilde{\nu}$, and let $p = \tilde{f}(q)$. Prove that

$$D\tilde{\nu}_q(w) \in T_p M \quad \text{for every } w \in \mathbb{R}^2.$$

Explain why $-D\tilde{\nu}_q \circ (D\tilde{f}_q)^{-1}$ therefore maps $T_p M$ to itself. Here the inverse of $D\tilde{f}_q$ is taken on its image $T_p M$. *Hint:* Differentiate $\langle \tilde{\nu}, \tilde{\nu} \rangle = 1$.

Solution. Differentiate the constant function $q \mapsto \langle \tilde{\nu}(q), \tilde{\nu}(q) \rangle = 1$ in the direction w . The product rule gives

$$0 = 2\langle \tilde{\nu}(q), D\tilde{\nu}_q(w) \rangle.$$

Thus $D\tilde{\nu}_q(w)$ is perpendicular to $\tilde{\nu}(q)$. Since $\tilde{\nu}(q)$ is a normal to the two-dimensional plane $T_p M$,

$$T_p M = \tilde{\nu}(q)^\perp.$$

This proves the claimed tangency. Notice that a derivative of a unit normal need not itself be normal; it is tangent.

Regularity means that $D\tilde{f}_q : \mathbb{R}^2 \rightarrow T_p M$ is a linear isomorphism. For each $X \in T_p M$, set $w = (D\tilde{f}_q)^{-1}X$. The result just proved implies

$$L_p X = -D\tilde{\nu}_q(w) \in T_p M.$$

Hence the shape operator is a linear map from $T_p M$ to itself, so it makes sense to discuss its eigenvalues and eigenvectors.