Lecture 1: Prerequisites from calculus

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The vector space $\mathbb{R}^n$ has its standard inner product

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i.$$

It determines the norm and distance

$$\|x\| = \sqrt{\langle x, x \rangle}, \qquad d(x, y) = \|x - y\|.$$

Two vectors $x, y \in \mathbb{R}^n$ are orthogonal if

$$\langle x, y \rangle = 0.$$

Lengths and angles come from the inner product. Differential geometry asks how these measurements behave on curved objects.

For $p \in \mathbb{R}^n$ and $r > 0$, the open ball centered at p is

$$B_r(p) = \{x \in \mathbb{R}^n : \|x - p\| < r\}.$$

Definition. A set $U \subset \mathbb{R}^n$ is open if every $p \in U$ is contained in some ball $B_r(p) \subset U$.

Let $U \subset \mathbb{R}^n$ be open. A map

$$F : U \longrightarrow \mathbb{R}^m$$

is C^k if its partial derivatives through order k exist and are continuous. It is smooth if it is C^k for every k.

The openness of U lets us perturb a point in every sufficiently small direction. This is what makes ordinary multivariable differentiation possible.

Definition. A map $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $p \in U$ if there is a linear map

$$dF_p : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

such that

$$F(p+h) = F(p) + dF_p(h) + r(h), \qquad \frac{\|r(h)\|}{\|h\|} \longrightarrow 0.$$

The map dF_p is called the derivative, or differential, of F at p.

Why it is unique. If two linear maps had this approximation property, their difference would satisfy

$$\|Lh\| = o(\|h\|).$$

Taking $h = tv$, dividing by $|t|$, and letting $t \to 0$ gives $Lv = 0$ for every v.

Write $F = (F_1, \ldots, F_m)$. In standard coordinates, dF_p is represented by the Jacobian matrix

$$J_p F = \begin{pmatrix} \dfrac{\partial F_1}{\partial x_1}(p) & \cdots & \dfrac{\partial F_1}{\partial x_n}(p) \\ \vdots & \ddots & \vdots \\ \dfrac{\partial F_m}{\partial x_1}(p) & \cdots & \dfrac{\partial F_m}{\partial x_n}(p) \end{pmatrix}.$$

Thus

$$dF_p(v) = J_p F\, v.$$

For a real-valued function f, this becomes

$$df_p(v) = \langle \nabla f(p), v \rangle.$$

The directional derivatives give the columns of the matrix: the j-th column is $dF_p(e_j)$.

Chain rule. Suppose

$$U \subset \mathbb{R}^n \xrightarrow{F} \mathbb{R}^m \xrightarrow{G} \mathbb{R}^k,$$

with F differentiable at p and G differentiable at $F(p)$. Then

$$d(G \circ F)_p = dG_{F(p)} \circ dF_p.$$

In coordinates,

$$J_p(G \circ F) = J_{F(p)}G\, J_pF.$$

Sketch of proof. Replace $F(p + h)$ by its linear approximation and then linearize G at $F(p)$. The linear terms compose. All remaining terms are small compared with $\|h\|$.

Definition. The rank of $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$ at p is

$$\operatorname{rank}_p F = \dim\bigl(\operatorname{im} dF_p\bigr) = \operatorname{rank}(J_p F).$$

The largest possible rank is

$$\min\{n, m\}.$$

Near a point where the rank is maximal and constant, a smooth map behaves like a linear map of the same rank. The inverse and implicit function theorems make this principle precise.

Later, rank will tell us when parametrized curves and surfaces are regular.

Inverse function theorem. Let $F : U \subset \mathbb{R}^n \to \mathbb{R}^n$ be C^1. If

$$\det J_p F \neq 0,$$

then there are neighborhoods V of p and W of $F(p)$ such that

$$F|_V : V \longrightarrow W$$

is a C^1 diffeomorphism. Moreover,

$$d(F^{-1})_{F(p)} = (dF_p)^{-1}.$$

Sketch of proof. After composing with an invertible linear map, assume $dF_p = I$. Near p, the map F is a small perturbation of the identity. A contraction argument solves $F(x) = y$ uniquely for nearby y.

Implicit function theorem. Let

$$F : U \subset \mathbb{R}^k \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$$

be C^1, with $F(a, b) = 0$. If the partial derivative

$$D_y F(a, b) : \mathbb{R}^m \longrightarrow \mathbb{R}^m$$

is invertible, then near (a, b) the equation $F(x, y) = 0$ has the form

$$y = g(x)$$

for a unique C^1 function g.

Sketch of proof. Apply the inverse function theorem to

$$\Phi(x, y) = (x, F(x, y)).$$

Its derivative is block triangular, with invertible diagonal blocks I and $D_y F(a, b)$.

Definition. A value $c \in \mathbb{R}^m$ is a regular value of $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$ if

$$dF_p : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

is surjective at every $p \in F^{-1}(c)$.

If c is a regular value, then $F^{-1}(c)$ is locally a smooth object of dimension $n - m$. Its tangent directions at p are

$$T_p F^{-1}(c) = \ker dF_p.$$

Sketch of proof. Surjectivity supplies an invertible $m \times m$ block in the Jacobian. The implicit function theorem then writes the level set locally as a graph over the remaining $n - m$ variables.

Consider

$$F : \mathbb{R}^3 \longrightarrow \mathbb{R}, \qquad F(x, y, z) = x^2 + y^2 + z^2.$$

Then

$$S^2 = F^{-1}(1).$$

At $p = (x, y, z) \in S^2$,

$$dF_p(v) = 2\langle p, v\rangle.$$

Since $p \neq 0$, the map $dF_p : \mathbb{R}^3 \to \mathbb{R}$ is surjective. Hence 1 is a regular value.

The tangent plane is therefore

$$T_pS^2 = \ker dF_p = \{v \in \mathbb{R}^3 : \langle p, v\rangle = 0\}.$$

Thus the familiar statement that the radius is normal to the sphere is exactly the equation $T_pS^2 = p^\perp$.

Consider an initial-value problem

$$x'(t) = X(t, x(t)), \qquad x(t_0) = x_0.$$

Existence and uniqueness principle. If X is sufficiently regular, then there is a unique local solution through (t_0, x_0).

Sketch of proof. Rewrite the equation as the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} X(s, x(s))\, ds.$$

On a sufficiently small interval, the right-hand side defines a contraction on a space of continuous paths. Its fixed point is the desired solution.

Later, the Frenet equations and the geodesic equations will both be treated as systems of ordinary differential equations.

The ideas from this lecture will be used in the following ways.

- A curve is a smooth map $c : I \longrightarrow \mathbb{R}^n$. Its derivative $c'(t)$ is its velocity.
- A parametrized surface is a smooth map $x : U \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ whose derivative has rank two.
- A regular equation $F(x) = c$ describes a smooth level set. Its tangent directions form $\ker dF$.
- The chain rule makes geometric definitions independent of coordinates.
- Existence and uniqueness for ODEs lets local geometric data determine curves.

The recurring theme is that nonlinear geometric objects become understandable after we replace them, locally, by their best linear approximations.