

Lecture 11: Hypersurfaces and the covariant derivative

An ***embedded regular hypersurface*** in $\mathbb{R}^{n+1}$ is a subset M which, near each point $p \in M$, is the regular zero set of one smooth real-valued function. There are an open neighborhood V of p and a smooth $F : V \rightarrow \mathbb{R}$ such that

$$M \cap V = F^{-1}(0), \quad dF_q \neq 0 \quad (q \in M \cap V).$$

Here $n \geq 1$. The implicit function theorem says that M has dimension n : locally, one coordinate is a smooth function of the other n coordinates.

The tangent space and a local unit normal are

$$T_p M = \ker dF_p = \nabla F(p)^\perp, \quad N(p) = \frac{\nabla F(p)}{\|\nabla F(p)\|}.$$

A hypersurface has codimension one. The familiar surfaces in $\mathbb{R}^3$ are the case $n = 2$.

Following Kühnel, a *regular hypersurface element* is a smooth map

$$f : U \longrightarrow \mathbb{R}^{n+1}, \quad U \subset \mathbb{R}^n \text{ open},$$

whose differential has rank n everywhere (so f is an immersion). Write

$$u = (u^1, \dots, u^n), \quad f_i = \frac{\partial f}{\partial u^i}, \quad f_{ij} = \frac{\partial^2 f}{\partial u^i \partial u^j}.$$

The vectors $f_1, \dots, f_n$ are linearly independent, and

$$T_{f(u)}f = \text{im } Df_u = \text{span}\{f_1(u), \dots, f_n(u)\}.$$

The notation f_i means a derivative of the whole vector-valued map, not its i -th component.

After restricting U near a point, f parametrizes an embedded regular hypersurface. Choose n ambient coordinates whose derivative minor is invertible. The inverse function theorem makes these coordinates local parameters, and the remaining coordinate is a graph.

An immersion need not be globally injective. All arguments below are local, on one such parametrized patch.

The basic examples work in every dimension.

An affine hyperplane is

$$M = \{x \in \mathbb{R}^{n+1} : \langle x, a \rangle = b\}, \quad a \neq 0.$$

Its normal $a/\|a\|$ is constant and $T_p M = a^\perp$ for every $p \in M$.

The sphere of radius $R > 0$ is

$$S_R^n = \{x \in \mathbb{R}^{n+1} : \|x\|^2 = R^2\}.$$

Taking $F(x) = \|x\|^2 - R^2$ gives

$$T_p S_R^n = p^\perp, \quad N(p) = p/R.$$

For a smooth function $H : U \subset \mathbb{R}^n \rightarrow \mathbb{R}$, its graph has parametrization and unit normal

$$f(u) = (u, H(u)), \quad v(u) = \frac{(-\nabla H(u), 1)}{\sqrt{1 + \|\nabla H(u)\|^2}}.$$

The first n rows of Df form the identity matrix, so this parametrization is regular.

The fundamental forms extend without changing their meaning. Choose a smooth local unit normal $\nu : U \rightarrow S^n$, and set

$$g_{ij} = \langle f_i, f_j \rangle, \quad h_{ij} = \langle f_{ij}, \nu \rangle.$$

The matrix $g = (g_{ij})$ is positive definite. It represents the first fundamental form $I(X, Y) = \langle X, Y \rangle$ in the tangent basis $(f_1, \dots, f_n)$.

The shape operator and second fundamental form are

$$L = -D\nu \circ (Df)^{-1}, \quad II(X, Y) = I(LX, Y).$$

In particular, $L(f_i) = -\nu_i$ and $II(f_i, f_j) = h_{ij}$. Since $f_{ij} = f_{ji}$, the form II is symmetric.

For tangent vectors with coordinate columns $a, b \in \mathbb{R}^n$,

$$I(X, Y) = a^\top g b, \quad II(X, Y) = a^\top h b.$$

Our goal is to differentiate tangent vector fields using only I . The construction will initially use the ambient space; Theorem 4.6 will show that the result is intrinsic.

A **vector field along** f is a smooth map $Y : U \rightarrow \mathbb{R}^{n+1}$, with $Y(u)$ regarded as a vector based at $f(u)$. It is **tangent** if

$$Y(u) \in \text{im } Df_u \quad \text{for every } u \in U.$$

Every tangent field has a unique expression

$$Y = \sum_{j=1}^n \eta^j f_j,$$

where the $\eta^j : U \rightarrow \mathbb{R}$ are smooth coefficient functions. Both the coefficients and the basis vectors can vary with u .

For $X = \sum_i \xi^i f_i$ and a smooth scalar function $\varphi : U \rightarrow \mathbb{R}$, define

$$X(\varphi) = \sum_{i=1}^n \xi^i \partial_i \varphi, \quad \partial_i = \frac{\partial}{\partial u^i}.$$

This is the rate of change of φ in direction X . In particular, $f_i(\varphi) = \partial_i \varphi$.

Superscripts on ξ^i, η^j, u^i are indices, not powers. We write all summation signs explicitly. All maps and fields in this lecture are smooth.

For a vector field Y on an open subset of $\mathbb{R}^{n+1}$, the **directional derivative** at p , in direction $X \in \mathbb{R}^{n+1}$, is

$$(D_X Y)_p = DY_p(X) = \left. \frac{d}{dt} Y(p + tX) \right|_{t=0}.$$

Here DY_p is the ordinary derivative, represented by the Jacobian matrix of Y .

One can replace the straight line by any smooth curve $c(t)$ with $c(0) = p$ and $\dot{c}(0) = X$. The chain rule gives

$$\left. \frac{d}{dt} Y(c(t)) \right|_{t=0} = DY_p(\dot{c}(0)) = DY_p(X).$$

Thus only the initial velocity of the curve matters. Its acceleration and its later motion are irrelevant.

For fixed Y , this expression is linear in the direction X . It requires the variation of Y near p , not just the single vector $Y(p)$.

Now suppose Y is defined only along the patch f . For $X \in T_{f(u)}f$, there is a unique $a \in \mathbb{R}^n$ with $Df_u(a) = X$. Define

$$(D_X Y)_{f(u)} = DY_u(a) = DY_u((Df_u)^{-1}X).$$

The curve $c(t) = f(u + ta)$ has $\dot{c}(0) = X$, and

$$(D_X Y)_{f(u)} = \left. \frac{d}{dt} Y(u + ta) \right|_{t=0}.$$

This definition does not require extending Y off the surface. If an ambient extension is used, the chain rule gives the same answer.

For $X = \sum_i \xi^i f_i$, the formula becomes

$$D_X Y = \sum_{i=1}^n \xi^i \partial_i Y.$$

In particular,

$$D_{f_i} Y = \partial_i Y, \quad D_{f_i} f_j = f_{ij}.$$

Even when X, Y are tangent, $D_X Y$ need not be tangent.

The unit cylinder provides the simplest warning. Take

$$f(u, v) = (\cos u, \sin u, v), \quad \nu = (\cos u, \sin u, 0).$$

The tangent field

$$Y = f_u = (-\sin u, \cos u, 0)$$

has unit length, but its derivative in its own direction is

$$D_{f_u} Y = f_{uu} = (-\cos u, -\sin u, 0) = -\nu.$$

The derivative is entirely normal to the cylinder.

Ambient differentiation sees the bending of the cylinder in $\mathbb{R}^3$. To measure change *within the tangent spaces*, we keep only the tangential component of this derivative.

In this example that component is zero. This does not say that Y is constant as a vector in $\mathbb{R}^3$; it says that its change has no tangential component.

For tangent fields X, Y , their ***covariant derivative*** is the tangential projection of the ambient derivative:

$$\nabla_X Y = (D_X Y)^\top = D_X Y - \langle D_X Y, \nu \rangle \nu.$$

Thus $\nabla_X Y$ is again tangent. At a point it depends only on the value of X , but on the local variation of Y .

The orthogonal projection

$$P^\top(V) = V - \langle V, \nu \rangle \nu$$

does not change if ν is replaced by $-\nu$. Consequently the covariant derivative does not require an orientation: local normal choices give the same answer on overlaps.

For a scalar function there is no projection to perform. We write simply

$$\nabla_X \varphi = D_X \varphi = X(\varphi).$$

The symbols D and ∇ distinguish ambient differentiation from tangential differentiation.

The discarded normal component is exactly the second fundamental form. Since Y is tangent,

$$\langle Y, \nu \rangle = 0.$$

Differentiate this identity in direction X :

$$\langle D_X Y, \nu \rangle + \langle Y, D_X \nu \rangle = 0.$$

Our sign convention for the shape operator is $D_X \nu = -LX$. Therefore

$$\langle D_X Y, \nu \rangle = \langle Y, LX \rangle = II(X, Y).$$

We obtain the decomposition

$$D_X Y = \nabla_X Y + II(X, Y)\nu.$$

The first term is tangential; the second is normal. The covariant derivative records tangential change, whereas II records bending in the ambient space.

For the unit cylinder, $II(f_u, f_u) = -1$, so the formula recovers $D_{f_u} f_u = 0 - \nu$.

The usual rules of differentiation pass to ∇ . For smooth scalar functions φ, ψ and tangent fields,

$$\nabla_{\varphi X + \psi Z} Y = \varphi \nabla_X Y + \psi \nabla_Z Y,$$

$$\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z,$$

and

$$\nabla_X (\varphi Y) = X(\varphi) Y + \varphi \nabla_X Y.$$

For the last identity, first differentiate in the ambient space and then project:

$$D_X (\varphi Y) = X(\varphi) Y + \varphi D_X Y.$$

The vector Y is already tangent, so its projection is itself.

Notice the asymmetry. Multiplication by a function can be pulled out of the *first* argument. In the second argument, there is an extra derivative term. Thus ∇ is not bilinear over smooth functions in both arguments.

The covariant derivative is *compatible with the metric*:

$$X(I(Y, Z)) = I(\nabla_X Y, Z) + I(Y, \nabla_X Z).$$

To see this, differentiate the Euclidean inner product:

$$X\langle Y, Z \rangle = \langle D_X Y, Z \rangle + \langle Y, D_X Z \rangle.$$

Use the tangential-normal decomposition in each term. Because Y and Z are tangent, the normal components contribute zero:

$$\langle D_X Y, Z \rangle = \langle \nabla_X Y, Z \rangle,$$

$$\langle Y, D_X Z \rangle = \langle Y, \nabla_X Z \rangle.$$

In particular,

$$X(I(Y, Y)) = 2I(\nabla_X Y, Y).$$

This identity will let us recover the derivative of the coordinate tangent basis from the metric coefficients.

The ***Jacobi bracket***, also called the ***Lie bracket***, of two tangent vector fields is

$$[X, Y] = D_X Y - D_Y X.$$

Although the two derivatives can have normal components, their difference is tangent. Indeed,

$$\langle D_X Y - D_Y X, \nu \rangle = II(X, Y) - II(Y, X) = 0.$$

Consequently,

$$[X, Y] = \nabla_X Y - \nabla_Y X.$$

This is the ***torsion-free*** identity. It does not say that $\nabla_X Y = \nabla_Y X$ for arbitrary fields: their difference is the bracket.

In particular, coordinate tangent fields commute:

$$[f_i, f_j] = f_{ij} - f_{ji} = 0, \quad \nabla_{f_i} f_j = \nabla_{f_j} f_i.$$

Here torsion-freeness is a property of the covariant derivative, not the torsion of a space curve.

The bracket can be computed without a normal vector or a metric.
Write

$$X = \sum_{i=1}^n \xi^i f_i, \quad Y = \sum_{j=1}^n \eta^j f_j.$$

The product rule gives

$$D_X Y = \sum_{i,j=1}^n \xi^i (\partial_i \eta^j) f_j + \sum_{i,j=1}^n \xi^i \eta^j f_{ij}.$$

In $D_X Y - D_Y X$, the second-derivative terms cancel because $f_{ij} = f_{ji}$.
Thus

$$[X, Y] = \sum_{j=1}^n \left(\sum_{i=1}^n (\xi^i \partial_i \eta^j - \eta^i \partial_i \xi^j) \right) f_j.$$

Equivalently, its j -th coefficient is

$$[X, Y]^j = X(\eta^j) - Y(\xi^j).$$

Coordinate fields commute, but fields with variable coefficients need not commute.

On the plane, let $e_x = (1, 0)$, $e_y = (0, 1)$, and choose

$$X = e_x, \quad Y = x e_y.$$

Then

$$D_X Y = \partial_x(x e_y) = e_y, \quad D_Y X = 0,$$

so

$$[X, Y] = e_y \neq 0.$$

Moving in the x -direction changes the speed prescribed by the field Y . Moving in the y -direction does not change the constant field X .

On the half-plane $x > 0$, these two fields are linearly independent, but they cannot both be coordinate basis fields for one coordinate system: coordinate basis fields must have zero bracket.

On a Euclidean plane, $D = \nabla$. This example also shows directly why covariant derivatives do not commute when the two vector fields are interchanged.

A vector field acts on scalar functions as a first-order differential operator. The bracket is the commutator of these operators:

$$[X, Y](\varphi) = X(Y(\varphi)) - Y(X(\varphi)).$$

To verify this, expand both sides in coordinates. The terms involving $\partial_i \partial_j \varphi$ cancel, leaving the coefficient formula from the preceding pages.

This characterization is intrinsic: it does not mention the ambient Euclidean space or the first fundamental form.

The bracket is skew-symmetric and satisfies the *Jacobi identity*:

$$[X, Y] = -[Y, X],$$

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

For a proof of the second identity, apply the left side to an arbitrary function. Expand the three commutators; every composition of three differential operators appears twice with opposite signs. A vector field is determined by its action on functions (in particular, on the coordinates).