Lecture 13: Parallel transport and intrinsic curvature

Let $c : [a, b] \to M \subset \mathbb{R}^{n+1}$ be a smooth curve. How can we move $v \in T_{c(a)}M$ along c while keeping it tangent and as nearly constant as possible?

For a tangent field $Y(t) \in T_{c(t)}M$, recall

$$\frac{\nabla Y}{dt} = (\dot{Y})^{\top}.$$

We call Y ***parallel along*** c if

$$\frac{\nabla Y}{dt} = 0.$$

Its ordinary derivative may be nonzero, but must be normal. With a unit normal $\nu(t)$ along c, differentiating $\langle Y, \nu \rangle = 0$ gives

$$\dot{Y} = -\langle Y, \dot{\nu} \rangle \nu.$$

This is a differential equation for the vector being moved, with initial condition $Y(a) = v$.

Choose a parametrization $f : U \to M$, and write

$$c(t) = f(u(t)), \qquad Y(t) = \sum_{j=1}^{n} \eta^j(t) f_j(u(t)), \qquad f_j = \partial_j f.$$

The formula from Lecture 12 reads

$$\frac{\nabla Y}{dt} = \sum_{k=1}^{n} \left(\dot{\eta}^k + \sum_{i,j=1}^{n} \Gamma_{ij}^k(u(t)) \dot{u}^i \eta^j \right) f_k.$$

Thus parallelism is the linear system

$$\dot{\eta} = -A(t)\eta, \qquad A^k{}_j(t) = \sum_{i=1}^{n} \dot{u}^i(t) \Gamma_{ij}^k(u(t)).$$

Theorem (Kühnel 4.10). For each initial tangent vector there is a unique parallel field along the prescribed curve.

Proof idea. A linear ODE with continuous coefficients has a unique solution throughout the curve's parameter interval. In overlapping charts the solutions agree by uniqueness, so the construction patches along the curve.

Define ***parallel transport*** by

$$P_c : T_{c(a)}M \longrightarrow T_{c(b)}M, \qquad P_c(v) = Y(b).$$

Parallel transport preserves the metric (4.11). For fields Y, Z along c, metric compatibility gives

$$\frac{d}{dt} I(Y, Z) = I\left(\frac{\nabla Y}{dt}, Z\right) + I\left(Y, \frac{\nabla Z}{dt}\right).$$

If both fields are parallel, this derivative is zero. Hence

$$I_{c(b)}(P_c v, P_c w) = I_{c(a)}(v, w).$$

In particular, lengths and angles are preserved.

Linearity and uniqueness of the ODE also imply

$$\begin{aligned} P_c(\alpha v + \beta w) &= \alpha P_c v + \beta P_c w, \\ P_{\bar{c}} &= P_c^{-1}, \\ P_{d*c} &= P_d \circ P_c. \end{aligned}$$

Here $\bar{c}$ traverses c backwards, and $d * c$ means first follow c, then d. For a piecewise smooth curve, transport is defined separately on each piece.

Changing to another increasing regular parameter multiplies $\nabla Y/dt$ by a nonzero scalar. It does not change transport. The path matters; its speed does not.

In the plane, parallel fields are simply constant ambient vectors. Their coordinate coefficients need not be constant.

For polar coordinates

$$f(r,\theta) = (r\cos\theta, r\sin\theta, 0), \qquad r > 0,$$

the nonzero Christoffel symbols are

$$\Gamma^r_{\theta\theta} = -r, \qquad \Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r}.$$

Along $c(t) = f(a,t)$, where $a > 0$ is constant, the parallel-transport equations become

$$\dot\eta^r - a\eta^\theta = 0, \qquad \dot\eta^\theta + \frac{1}{a}\eta^r = 0.$$

For $Y(0) = (1,0,0)$, their solution is

$$\eta^r(t) = \cos t, \qquad \eta^\theta(t) = -\frac{\sin t}{a}.$$

Indeed,

$$Y(t) = \cos t\, f_r - \frac{\sin t}{a} f_\theta = (1,0,0).$$

The frame turns, while the parallel vector stays fixed.

Consider the unit sphere and its great-circle arc

$$c(s) = (\cos s, \sin s, 0), \qquad 0 \le s \le \frac{\pi}{2}.$$

This parameter is arclength. Put

$$T(s) = c'(s) = (-\sin s, \cos s, 0), \qquad B = (0, 0, 1).$$

These form an orthonormal basis of $T_{c(s)}S^2$. Moreover,

$$T'(s) = -c(s), \qquad \frac{dB}{ds} = 0.$$

Both derivatives have zero tangential part, so T and B are parallel along the arc.

Consequently, the transport of $v = \alpha T(0) + \beta B$ is given explicitly by

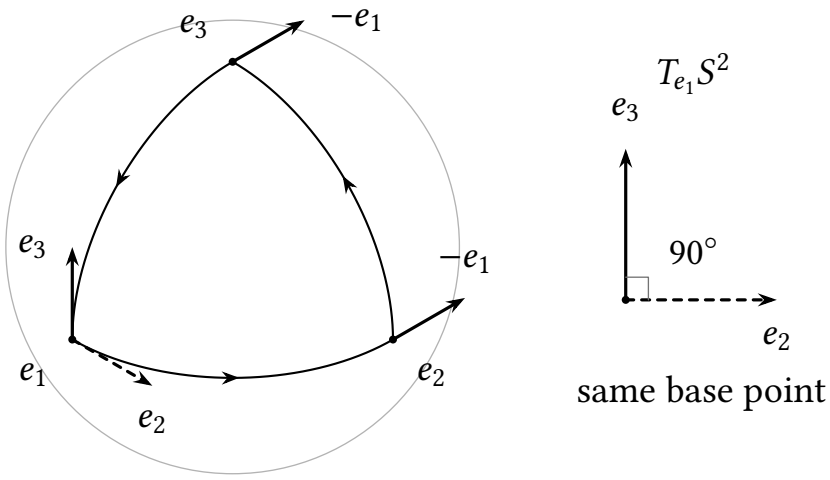
$$Y(s) = \alpha T(s) + \beta B.$$

For example, $P_c(e_2) = -e_1$.

The ambient vector changes direction, but its components in this parallel orthonormal frame remain constant.

Transport around a closed curve need not return a vector to itself. On the unit sphere, follow the quarter-circle arcs

$$e_1 \longrightarrow e_2 \longrightarrow e_3 \longrightarrow e_1.$$



Start with the dashed vector e_2 at e_1. Along the first arc it becomes $-e_1$; along the second arc it stays constant. On the last arc,

$$c(s) = \sin s\, e_1 + \cos s\, e_3, \qquad Y(s) = -c'(s), \qquad 0 \le s \le \frac{\pi}{2}.$$

Thus Y goes from $-e_1$ to the solid vector e_3 at the original base point:

$$P_c(e_2) = e_3.$$

This quarter-turn in $T_{e_1}S^2$ is the ***holonomy*** of the loop. Transport depends on the path, not just on its endpoints.

A curve is an ***affinely parametrized geodesic*** if its own velocity is parallel:

$$\nabla_{\dot{c}}\dot{c} = 0.$$

For an embedded hypersurface this means

$$(\ddot{c})^{\top} = 0, \qquad \ddot{c} = II(\dot{c}, \dot{c})\nu.$$

The particle may accelerate in ambient space, but has no tangential acceleration.

Every such geodesic has constant speed, since

$$\frac{d}{dt} I(\dot{c}, \dot{c}) = 2I(\nabla_{\dot{c}}\dot{c}, \dot{c}) = 0.$$

For a unit-speed geodesic we instead use $c = c(s)$ and write

$$\nabla_{c'} c' = 0.$$

Lines in the plane are geodesics. The great circle from the previous example is also a geodesic: $c'' = -c$ is normal to the sphere. This illustrates why a geodesic need not look straight in the surrounding space.

Setting $\eta^j = \dot{u}^j$ in the equation for a parallel field gives the ***geodesic equations***:

$$\ddot{u}^k + \sum_{i,j=1}^{n} \Gamma_{ij}^k(u)\dot{u}^i\dot{u}^j = 0, \qquad k = 1, \dots, n.$$

Here the path $u(t)$ is unknown, unlike the prescribed path in the parallel-transport problem.

Theorem (Kühnel 4.12). For $p \in M$ and $v \in T_pM$, there is a unique local geodesic with $c(0) = p$ and $\dot{c}(0) = v$. If $\|v\| = 1$, it is parametrized by arclength.

Proof idea. Set $w^k = \dot{u}^k$. The equations become

$$\dot{u}^k = w^k, \qquad \dot{w}^k = -\sum_{i,j=1}^{n} \Gamma_{ij}^k(u) w^i w^j.$$

This is a smooth first-order system for (u, w), so the usual local existence and uniqueness theorem applies. Constant speed proves the unit-speed assertion.

The conclusion is local; existence for all time requires an additional hypothesis.

On the circular cylinder of radius $a > 0$, use

$$f(u, z) = \big(a\cos(u/a), a\sin(u/a), z\big).$$

Locally this parametrizes the cylinder by distance around it and height. Since

$$g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

all Christoffel symbols vanish. The geodesic equations are

$$\ddot{u} = 0, \qquad \ddot{z} = 0.$$

Thus every geodesic has the form

$$c(t) = f(u_0 + \alpha t, z_0 + \beta t).$$

These are vertical lines, horizontal circles, or helices, according to α and β. The speed is $\sqrt{\alpha^2 + \beta^2}$.

Unrolling the cylinder turns these curves into straight lines. Intrinsically the cylinder is locally flat, despite its bending in $\mathbb{R}^3$.

Parallel transport is equally simple here: the coefficients of a parallel vector in the frame f_u, f_z are constant.

For a surface of rotation, choose arclength u along the profile curve:

$$f(u,\theta) = \big(r(u)\cos\theta, r(u)\sin\theta, z(u)\big), \qquad r > 0, \qquad r_u^2 + z_u^2 = 1.$$

Then

$$I = du^2 + r(u)^2 d\theta^2,$$

and the only possibly nonzero Christoffel symbols are

$$\Gamma^u_{\theta\theta} = -r r_u, \qquad \Gamma^\theta_{u\theta} = \Gamma^\theta_{\theta u} = \frac{r_u}{r}.$$

The geodesic equations become

$$\ddot{u} - r r_u \dot{\theta}^2 = 0, \qquad \ddot{\theta} + 2\frac{r_u}{r}\dot{u}\dot{\theta} = 0.$$

A meridian $\theta =$ constant, with u affine, is therefore a geodesic. A nonconstant parallel $u = u_0$, with θ affine, is a geodesic precisely when $r_u(u_0) = 0$.

For the sphere, $r(u) = a\sin(u/a)$: meridians are geodesics, and the only geodesic parallel is the equator.

The image of a geodesic and an affinely parametrized geodesic are not quite the same notion.

For a regular change of parameter $t = t(r)$, set $\widetilde{c}(r) = c(t(r))$. The chain and product rules give

$$\frac{\nabla}{dr}\frac{d\widetilde{c}}{dr} = \left(\frac{dt}{dr}\right)^2 \nabla_{\dot{c}}\dot{c} + \frac{d^2t}{dr^2}\dot{c}.$$

If c is a nonconstant affine geodesic, the new parametrization is affine exactly when $d^2t/dr^2 = 0$.

More generally, a regular curve can be reparametrized as a geodesic if and only if

$$\nabla_{\dot{c}}\dot{c} = \lambda(t)\dot{c}$$

for some function λ.

To see the converse, put $v = \|\dot{c}\|$. Metric compatibility gives $\dot{v}/v = \lambda$. Arclength then removes precisely this tangential change of speed:

$$\nabla_{c'}c' = \frac{1}{v^2}\left(\nabla_{\dot{c}}\dot{c} - \frac{\dot{v}}{v}\dot{c}\right) = 0.$$

Shortest paths are geodesics (Kühnel 4.13). Suppose a smooth regular curve minimizes length among curves with its endpoints. Parametrize it by arclength, $c : [0, \ell] \to M$.

Consider a smooth variation $C(s, \varepsilon)$ in M such that

$$C(s, 0) = c(s),$$
$$C(0, \varepsilon) = c(0), \qquad C(\ell, \varepsilon) = c(\ell).$$

Its ***variation field*** is

$$V(s) = \left.\frac{\partial C}{\partial \varepsilon}\right|_{\varepsilon=0}, \qquad V(0) = V(\ell) = 0.$$

Writing $\mathscr{L}(\varepsilon)$ for the length of the varied curve, differentiation under the integral gives

$$\mathscr{L}'(0) = \int_0^\ell \left\langle \left.\frac{\partial^2 C}{\partial s \partial \varepsilon}\right|_{\varepsilon=0}, c' \right\rangle ds = \int_0^\ell I(\nabla_{c'} V, c')\, ds.$$

Only the original curve has unit speed; the nearby curves need not be arclength parametrized.

Metric compatibility and integration by parts give

$$\begin{aligned}\mathscr{L}'(0) &= \big[I(V, c')\big]_0^\ell - \int_0^\ell I(V, \nabla_{c'} c')\, ds \\ &= -\int_0^\ell I(V, \nabla_{c'} c')\, ds.\end{aligned}$$

This is the ***first variation of length*** for fixed endpoints and a unit-speed base curve.

For a minimizing curve, $\mathscr{L}'(0) = 0$ for every variation. If $A = \nabla_{c'} c'$ were nonzero somewhere, choose a nonnegative, nonzero bump function χ, supported in a small coordinate interval where $A \neq 0$, and take $V = \chi A$. Then

$$\mathscr{L}'(0) = -\int_0^\ell \chi \|A\|^2\, ds < 0,$$

a contradiction. Such a field is realized by varying the coordinates of c and leaving the curve fixed outside that interval. Hence $\nabla_{c'} c' = 0$.

The converse is not globally true. A great-circle arc longer than a semicircle but shorter than a full circle is a geodesic; the complementary arc joins its endpoints more briefly.

The first fundamental form determines ∇, parallel transport, and geodesics. Can the first and second fundamental forms be prescribed independently?

Recall the Gauss and Weingarten formulas:

$$D_X Y = \nabla_X Y + II(X, Y)\nu, \qquad D_X \nu = -LX,$$

where $II(X, Y) = I(LX, Y)$. In coordinates they are

$$f_{ij} = \sum_k \Gamma_{ij}^k f_k + h_{ij}\nu, \qquad \partial_i \nu = -\sum_{j,k} h_{ij} g^{jk} f_k.$$

Here $g_{ij} = I(f_i, f_j)$, $h_{ij} = II(f_i, f_j)$, and $(g^{ij}) = (g_{ij})^{-1}$.

The moving frame is $f_1, \ldots, f_n, \nu$. Its derivatives are described by the equations above. A solution must have commuting mixed partial derivatives, for example

$$\partial_k f_{ij} = \partial_j f_{ik}.$$

This forces compatibility conditions on I and II. We derive them without expanding every coordinate index.

For ordinary derivatives, the Jacobi bracket accounts for the failure of two directions to commute (Kühnel 4.17):

$$D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z = 0.$$

In coordinates, terms with second derivatives of Z cancel by equality of mixed partials; the remaining terms are exactly $D_{[X,Y]}Z$. This also applies to ambient vector-valued fields along a surface.

Covariant derivatives need not satisfy the same identity. Define the ***curvature operator*** by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The bracket correction is essential: it removes the failure caused merely by using noncommuting fields.

The formula involves only ∇ and the bracket. Consequently it is determined by the first fundamental form. On the next page we will see why its value at a point depends only on the three vectors at that point, not on how they are extended to fields.

The Gauss equation (Kühnel 4.18). For tangent vector fields on a hypersurface,

$$R(X,Y)Z = II(Y,Z)LX - II(X,Z)LY.$$

Proof. Apply the Gauss formula twice. Since $D_X\nu = -LX$, the tangential part is

$$(D_X D_Y Z)^\top = \nabla_X \nabla_Y Z - II(Y,Z)LX.$$

Swap X, Y, subtract, and use $(D_{[X,Y]}Z)^\top = \nabla_{[X,Y]}Z$. The ambient commutator identity becomes

$$0 = R(X,Y)Z - II(Y,Z)LX + II(X,Z)LY,$$

which proves the assertion.

The right-hand side is pointwise and trilinear in X, Y, Z. Thus so is the left-hand side. This is why R is called the ***curvature tensor*** (4.19).

The same quantity has an intrinsic description through ∇ and an extrinsic description through L.

The normal component of the ambient commutator identity gives the other compatibility condition. Its scalar coefficient is

$$\begin{aligned} 0 = {} & X\big(II(Y,Z)\big) - Y\big(II(X,Z)\big) \\ & + II(X, \nabla_Y Z) - II(Y, \nabla_X Z) \\ & - II([X,Y], Z). \end{aligned}$$

Replace $II(U,V)$ by $I(LU,V)$ and differentiate using metric compatibility. The terms involving derivatives of Z cancel, leaving

$$I\big(\nabla_X(LY) - \nabla_Y(LX) - L[X,Y], Z\big) = 0.$$

Since Z is arbitrary, the vector in parentheses is zero.

Define $(\nabla_X L)Y = \nabla_X(LY) - L(\nabla_X Y)$. Using $[X,Y] = \nabla_X Y - \nabla_Y X$, we obtain the ***Codazzi–Mainardi equation***:

$$(\nabla_X L)Y = (\nabla_Y L)X.$$

Together with Gauss's equation, this expresses the compatibility of the moving-frame equations (4.15, 4.18).

Gauss's Theorema Egregium (4.16, 4.20). The Gaussian curvature of a surface is determined by its first fundamental form.

Proof. At a point choose an orthonormal basis e_1, e_2. Since L is self-adjoint, its matrix is

$$[L] = \begin{pmatrix} a & b \\ b & d \end{pmatrix}, \qquad K = \det L = ad - b^2.$$

The Gauss equation gives

$$\begin{aligned} I\big(R(e_1, e_2)e_2, e_1\big) &= II(e_2, e_2)II(e_1, e_1) - II(e_1, e_2)^2 \\ &= ad - b^2 = K. \end{aligned}$$

But R is determined by the metric, so the same is true of K.

For any tangent basis X, Y, the equivalent formula is

$$K = \frac{I(R(X, Y)Y, X)}{I(X, X)I(Y, Y) - I(X, Y)^2}.$$

The denominator is the squared area of the parallelogram spanned by X, Y.

We can now compute Gaussian curvature without choosing a normal. For

$$I = du^2 + r(u)^2 d\theta^2,$$

write $U = f_u$, $V = f_\theta$. The Christoffel symbols computed earlier say

$$\nabla_U U = 0, \quad \nabla_U V = \nabla_V U = \frac{r_u}{r} V, \quad \nabla_V V = -r r_u U.$$

Since $[U, V] = 0$ and $V(r) = 0$,

$$\begin{aligned} R(U,V)V &= \nabla_U(-r r_u U) - \nabla_V\left(\frac{r_u}{r} V\right) \\ &= -(r_u^2 + r r_{uu})U + r_u^2 U \\ &= -r r_{uu} U. \end{aligned}$$

The area denominator is r^2, so

$$K = -\frac{r_{uu}}{r}.$$

For a cylinder, $r = a$, giving $K = 0$. For a sphere, $r(u) = a\sin(u/a)$, giving $K = 1/a^2$.

A ***local isometry*** is a local diffeomorphism $\Phi : M \to \widetilde{M}$ satisfying

$$\widetilde{I}_{\Phi(p)}(d\Phi_p v, d\Phi_p w) = I_p(v, w).$$

It preserves the metric, so it preserves covariant differentiation, parallel transport, geodesics, and now Gaussian curvature:

$$\widetilde{K} \circ \Phi = K.$$

The plane and cylinder are locally isometric: both have metric $du^2 + dz^2$ in the coordinates used above. Their Gaussian curvatures are both zero, but their shape operators are different. With the outward cylinder normal,

$$L_{\text{plane}} = 0, \qquad [L_{\text{cylinder}}] = \begin{pmatrix} -1/a & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus mean curvature is not intrinsic: it is zero on the plane and $-1/(2a)$ on this oriented cylinder.

By contrast, no open piece of a sphere is isometric to an open piece of the plane: $1/a^2 \neq 0$. A sphere cannot be flattened without changing lengths, even locally.