Lecture 14: Geodesics and intrinsic curvature

Recall that an affinely parametrized ***geodesic*** satisfies

$$\nabla_{\dot{c}}\dot{c} = (\ddot{c})^{\top} = 0.$$

The acceleration must be normal to the surface, not necessarily zero in the surrounding space.

Consider a circle in the plane $M = \{z = 0\}$:

$$c(s) = \big(a\cos(s/a), a\sin(s/a), 0\big), \qquad a > 0.$$

It has unit speed, but

$$c''(s) = -\frac{1}{a}\big(\cos(s/a), \sin(s/a), 0\big).$$

This vector is tangent to the plane. Therefore

$$\nabla_{c'}c' = c'' \neq 0.$$

The circle is not a geodesic of the plane. Constant speed does not suffice: the curve is turning within the surface.

A circle on a sphere need not be a geodesic either. Fix $-1 < h < 1$, put $r = \sqrt{1-h^2}$, and consider the latitude

$$c(s) = \big(r\cos(s/r), r\sin(s/r), h\big) \in S^2.$$

Since $\|c'(s)\| = 1$, this is an arclength parameter. The outward unit normal is $c(s)$, and

$$c''(s) = -\frac{1}{r}\big(\cos(s/r), \sin(s/r), 0\big), \qquad \langle c'', c\rangle = -1.$$

Subtracting the normal component gives

$$\begin{aligned}\nabla_{c'}c' &= c'' - \langle c'', c\rangle c = c'' + c \\ &= \left(-\frac{h^2}{r}\cos(s/r), -\frac{h^2}{r}\sin(s/r), h\right).\end{aligned}$$

In particular,

$$\|\nabla_{c'}c'\| = \frac{|h|}{r}.$$

Thus a non-equatorial latitude is not a geodesic. Its acceleration has a nonzero component tangent to the sphere. The equator $h = 0$ is the exception.

On the unit cylinder $x^2 + y^2 = 1$, consider

$$c(t) = (\cos t, \sin t, t^2).$$

Its height changes quadratically as it winds around the cylinder. The velocity, acceleration, and unit normal are

$$\begin{aligned}
\dot{c}(t) &= (-\sin t, \cos t, 2t), \\
\ddot{c}(t) &= (-\cos t, -\sin t, 2), \\
v(t) &= (\cos t, \sin t, 0).
\end{aligned}$$

Since $\langle \ddot{c}, v \rangle = -1$,

$$\nabla_{\dot{c}} \dot{c} = \ddot{c} - \langle \ddot{c}, v \rangle v = (0, 0, 2).$$

There is a vertical tangential acceleration, so this curve is not a geodesic.

This is not merely a poor choice of speed. The vector $(0, 0, 2)$ is never parallel to $\dot{c}$, whose horizontal component is nonzero. We will see that a change of parameter can remove only an acceleration parallel to the velocity.

Unrolling the cylinder turns this path into the parabola $z = t^2$, rather than a straight line.

There is a different reason a parametrized curve can fail the geodesic equation. In the plane, let

$$c(t) = (t^2, 0, 0), \qquad t > 0.$$

Then

$$\dot{c} = (2t, 0, 0), \qquad \nabla_{\dot{c}}\dot{c} = (2, 0, 0) = \frac{1}{t}\dot{c}.$$

Thus $c(t)$ is not an affinely parametrized geodesic, even though its image is a straight ray.

Here the acceleration changes the speed, not the direction. With the arclength coordinate $s = t^2$, the same path is

$$\gamma(s) = (s, 0, 0), \qquad \|\gamma'(s)\| = 1, \qquad \nabla_{\gamma'}\gamma' = 0.$$

It is useful to distinguish a ***geodesic path*** from its parametrization. A geodesic path admits a parameter for which the covariant acceleration vanishes; an arbitrary parameter need not have this property.

The circle and latitude examples were already at unit speed, so their failure cannot be repaired this way.

The image of a geodesic and an affinely parametrized geodesic are not quite the same notion.

For a regular change of parameter $t = t(r)$, set $\widetilde{c}(r) = c(t(r))$. The chain and product rules give

$$\frac{\nabla}{dr}\frac{d\widetilde{c}}{dr} = \left(\frac{dt}{dr}\right)^2 \nabla_{\dot{c}}\dot{c} + \frac{d^2t}{dr^2}\dot{c}.$$

If c is a nonconstant affine geodesic, the new parametrization is affine exactly when $d^2t/dr^2 = 0$.

More generally, a regular curve can be reparametrized as a geodesic if and only if

$$\nabla_{\dot{c}}\dot{c} = \lambda(t)\dot{c}$$

for some function λ.

To see the converse, put $v = \|\dot{c}\|$. Metric compatibility gives $\dot{v}/v = \lambda$. Arclength then removes precisely this tangential change of speed:

$$\nabla_{c'}c' = \frac{1}{v^2}\left(\nabla_{\dot{c}}\dot{c} - \frac{\dot{v}}{v}\dot{c}\right) = 0.$$

In local coordinates, write $c(t) = f(u(t))$ and $f_i = \partial_i f$. Differentiating $\dot{c} = \sum_i \dot{u}^i f_i$ covariantly gives the ***geodesic equations***:

$$\ddot{u}^k + \sum_{i,j=1}^{n} \Gamma_{ij}^k(u)\dot{u}^i\dot{u}^j = 0, \qquad k = 1, \dots, n.$$

The path $u(t)$ is now unknown, unlike the prescribed path in the parallel-transport problem.

Theorem (Kühnel 4.12). For $p \in M$ and $v \in T_pM$, there is a unique local geodesic with $c(0) = p$ and $\dot{c}(0) = v$.

Proof idea. Set $w^k = \dot{u}^k$. Then

$$\dot{u}^k = w^k, \qquad \dot{w}^k = -\sum_{i,j=1}^{n} \Gamma_{ij}^k(u) w^i w^j.$$

Apply the local existence and uniqueness theorem to this smooth first-order system for (u, w).

The speed is constant because

$$\frac{d}{dt} I(\dot{c}, \dot{c}) = 2I(\nabla_{\dot{c}}\dot{c}, \dot{c}) = 0.$$

Hence unit initial velocity gives an arclength parameter. Existence for all time requires an additional hypothesis.

Let $S_a^2 = \{x \in \mathbb{R}^3 : \|x\| = a\}$, with $a > 0$. Choose $p \in S_a^2$ and $w \in T_p S_a^2$ with $\|w\| = 1$. Define

$$c(s) = \cos(s/a)\, p + a \sin(s/a)\, w.$$

Since $\|p\| = a$ and $\langle p, w \rangle = 0$, we have

$$\|c(s)\| = a, \qquad \|c'(s)\| = 1, \qquad c''(s) = -\frac{1}{a^2} c(s).$$

The acceleration is normal to the sphere. Thus c is a geodesic with $c(0) = p$ and $c'(0) = w$. Its image is the great circle in the plane spanned by p, w.

Conversely, a unit-speed geodesic on S_a^2 has $c'' = \lambda c$. Differentiating $\langle c, c \rangle = a^2$ twice gives

$$\langle c, c'' \rangle = -\langle c', c' \rangle = -1, \qquad \lambda = -\frac{1}{a^2}.$$

Solving this linear ODE gives the formula above.

Thus the unit-speed geodesics are exactly great circles traversed at unit speed. An arc may be a geodesic even when it is too long to be a shortest path.

On the circular cylinder of radius $a > 0$, use

$$f(u, z) = \big(a\cos(u/a), a\sin(u/a), z\big).$$

Locally this parametrizes the cylinder by distance around it and height. Since

$$g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

all Christoffel symbols vanish. The geodesic equations are

$$\ddot{u} = 0, \qquad \ddot{z} = 0.$$

Thus every geodesic has the form

$$c(t) = f(u_0 + \alpha t, z_0 + \beta t).$$

For nonzero velocity, these are vertical lines, horizontal circles, or helices, depending on α, β. The speed is $\sqrt{\alpha^2 + \beta^2}$.

Unrolling the cylinder turns these curves into straight lines. Intrinsically the cylinder is locally flat, despite its bending in $\mathbb{R}^3$.

Parallel transport is equally simple here: the coefficients of a parallel vector in the frame f_u, f_z are constant.

A ***surface of revolution*** is obtained by rotating a plane profile $(r(u), z(u))$ about the z-axis:

$$f(u, \theta) = \big(r(u)\cos\theta, r(u)\sin\theta, z(u)\big).$$

Assume $r > 0$ and $r_u^2 + z_u^2 > 0$. Fixing u gives a circle, called a ***parallel***; fixing θ gives a ***meridian***. The angle is a local coordinate.

To compute the first fundamental form, differentiate:

$$f_u = (r_u \cos\theta, r_u \sin\theta, z_u),$$
$$f_\theta = (-r\sin\theta, r\cos\theta, 0).$$

Taking inner products gives

$$g_{ij} = \langle f_i, f_j \rangle, \qquad g = \begin{pmatrix} r_u^2 + z_u^2 & 0 \\ 0 & r^2 \end{pmatrix}.$$

Thus

$$I = (r_u^2 + z_u^2)\, du^2 + r^2\, d\theta^2.$$

The coordinate directions are perpendicular. Moreover,

$$\|f_u \times f_\theta\|^2 = r^2(r_u^2 + z_u^2) > 0,$$

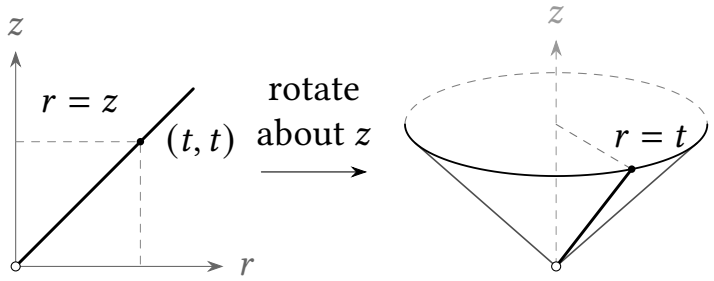
so this is a regular surface element.

If u is arclength along the profile, then $r_u^2 + z_u^2 = 1$, and the metric simplifies to

$$I = du^2 + r(u)^2\, d\theta^2.$$

For example, rotate the ray $r = z = t > 0$. This gives the lateral surface of a cone:

$$f(t, \theta) = (t \cos\theta, t \sin\theta, t).$$



Here $r_t = z_t = 1$, so the general formula gives

$$g = \begin{pmatrix} 2 & 0 \\ 0 & t^2 \end{pmatrix}, \qquad I = 2\,dt^2 + t^2\,d\theta^2.$$

The parameter t is not arclength along a meridian: the profile has speed $\sqrt{2}$. With $u = \sqrt{2}\,t$,

$$r(u) = z(u) = \frac{u}{\sqrt{2}}, \qquad I = du^2 + \frac{u^2}{2}\,d\theta^2.$$

Thus u measures slant distance, while t measures both radius and height. The apex is excluded: the cone is not smooth there.

For a surface of rotation, choose arclength u along the profile curve:

$$f(u,\theta) = \big(r(u)\cos\theta, r(u)\sin\theta, z(u)\big), \qquad r > 0, \qquad r_u^2 + z_u^2 = 1.$$

Then

$$I = du^2 + r(u)^2 d\theta^2,$$

and the only possibly nonzero Christoffel symbols are

$$\Gamma^u_{\theta\theta} = -r r_u, \qquad \Gamma^\theta_{u\theta} = \Gamma^\theta_{\theta u} = \frac{r_u}{r}.$$

The geodesic equations become

$$\ddot{u} - r r_u \dot{\theta}^2 = 0, \qquad \ddot{\theta} + 2\frac{r_u}{r}\dot{u}\dot{\theta} = 0.$$

A meridian $\theta =$ constant, with u affine, is therefore a geodesic. A nonconstant parallel $u = u_0$, with θ affine, is a geodesic precisely when $r_u(u_0) = 0$.

For the sphere, $r(u) = a\sin(u/a)$: meridians are geodesics, and the only geodesic parallel is the equator.

Shortest paths are geodesics (Kühnel 4.13). Suppose a smooth regular curve minimizes length among curves with its endpoints. Parametrize it by arclength, $c : [0, \ell] \to M$.

Consider a smooth variation $C(s, \varepsilon)$ in M such that

$$C(s, 0) = c(s),$$
$$C(0, \varepsilon) = c(0), \qquad C(\ell, \varepsilon) = c(\ell).$$

Its ***variation field*** is

$$V(s) = \left.\frac{\partial C}{\partial \varepsilon}\right|_{\varepsilon=0}, \qquad V(0) = V(\ell) = 0.$$

Writing $\mathscr{L}(\varepsilon)$ for the length of the varied curve, differentiation under the integral gives

$$\mathscr{L}'(0) = \int_0^\ell \left\langle \left.\frac{\partial^2 C}{\partial s \partial \varepsilon}\right|_{\varepsilon=0}, c' \right\rangle ds = \int_0^\ell I(\nabla_{c'} V, c')\, ds.$$

Only the original curve has unit speed; the nearby curves need not be arclength parametrized.

Metric compatibility and integration by parts give

$$\begin{aligned}\mathscr{L}'(0) &= \big[I(V, c')\big]_0^\ell - \int_0^\ell I(V, \nabla_{c'} c')\, ds \\ &= -\int_0^\ell I(V, \nabla_{c'} c')\, ds.\end{aligned}$$

This is the ***first variation of length*** for fixed endpoints and a unit-speed base curve.

For a minimizing curve, $\mathscr{L}'(0) = 0$ for every variation. If $A = \nabla_{c'} c'$ were nonzero somewhere, choose a nonnegative, nonzero bump function χ, supported in a small coordinate interval where $A \neq 0$, and take $V = \chi A$. Then

$$\mathscr{L}'(0) = -\int_0^\ell \chi \|A\|^2\, ds < 0,$$

a contradiction. Such a field is realized by varying the coordinates of c and leaving the curve fixed outside that interval. Hence $\nabla_{c'} c' = 0$.

The converse is not globally true. A great-circle arc longer than a semicircle but shorter than a full circle is a geodesic; the complementary arc joins its endpoints more briefly.

The first fundamental form determines ∇, parallel transport, and geodesics. Can the first and second fundamental forms be prescribed independently?

Recall the Gauss and Weingarten formulas:

$$D_X Y = \nabla_X Y + II(X, Y)\nu, \qquad D_X \nu = -LX,$$

where $II(X, Y) = I(LX, Y)$. In coordinates they are

$$f_{ij} = \sum_k \Gamma_{ij}^k f_k + h_{ij}\nu, \qquad \partial_i \nu = -\sum_{j,k} h_{ij} g^{jk} f_k.$$

Here $g_{ij} = I(f_i, f_j)$, $h_{ij} = II(f_i, f_j)$, and $(g^{ij}) = (g_{ij})^{-1}$.

The moving frame is $f_1, \ldots, f_n, \nu$. Its derivatives are described by the equations above. A solution must have commuting mixed partial derivatives, for example

$$\partial_k f_{ij} = \partial_j f_{ik}.$$

This forces compatibility conditions on I and II. We derive them without expanding every coordinate index.

For ordinary derivatives, the Jacobi bracket accounts for the failure of two directions to commute (Kühnel 4.17):

$$D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z = 0.$$

In coordinates, terms with second derivatives of Z cancel by equality of mixed partials; the remaining terms are exactly $D_{[X,Y]}Z$. This also applies to ambient vector-valued fields along a surface.

Covariant derivatives need not satisfy the same identity. Define the ***curvature operator*** by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The bracket correction is essential: it removes the failure caused merely by using noncommuting fields.

The formula involves only ∇ and the bracket. Consequently it is determined by the first fundamental form. On the next page we will see why its value at a point depends only on the three vectors at that point, not on how they are extended to fields.

The Gauss equation (Kühnel 4.18). For tangent vector fields on a hypersurface,

$$R(X,Y)Z = II(Y,Z)LX - II(X,Z)LY.$$

Proof. Apply the Gauss formula twice. Since $D_X\nu = -LX$, the tangential part is

$$(D_X D_Y Z)^\top = \nabla_X \nabla_Y Z - II(Y,Z)LX.$$

Swap X, Y, subtract, and use $(D_{[X,Y]}Z)^\top = \nabla_{[X,Y]}Z$. The ambient commutator identity becomes

$$0 = R(X,Y)Z - II(Y,Z)LX + II(X,Z)LY,$$

which proves the assertion.

The right-hand side is pointwise and trilinear in X, Y, Z. Thus so is the left-hand side. This is why R is called the ***curvature tensor*** (4.19).

The same quantity has an intrinsic description through ∇ and an extrinsic description through L.

The normal component of the ambient commutator identity gives the other compatibility condition. Its scalar coefficient is

$$\begin{aligned} 0 = {} & X\big(II(Y,Z)\big) - Y\big(II(X,Z)\big) \\ & + II(X, \nabla_Y Z) - II(Y, \nabla_X Z) \\ & - II([X,Y], Z). \end{aligned}$$

Replace $II(U,V)$ by $I(LU,V)$ and differentiate using metric compatibility. The terms involving derivatives of Z cancel, leaving

$$I\big(\nabla_X(LY) - \nabla_Y(LX) - L[X,Y], Z\big) = 0.$$

Since Z is arbitrary, the vector in parentheses is zero.

Define $(\nabla_X L)Y = \nabla_X(LY) - L(\nabla_X Y)$. Using $[X,Y] = \nabla_X Y - \nabla_Y X$, we obtain the ***Codazzi–Mainardi equation***:

$$(\nabla_X L)Y = (\nabla_Y L)X.$$

Together with Gauss's equation, this expresses the compatibility of the moving-frame equations (4.15, 4.18).

Gauss's Theorema Egregium (4.16, 4.20). The Gaussian curvature of a surface is determined by its first fundamental form.

Proof. At a point choose an orthonormal basis e_1, e_2. Since L is self-adjoint, its matrix is

$$[L] = \begin{pmatrix} a & b \\ b & d \end{pmatrix}, \qquad K = \det L = ad - b^2.$$

The Gauss equation gives

$$\begin{aligned} I\big(R(e_1, e_2)e_2, e_1\big) &= II(e_2, e_2)II(e_1, e_1) - II(e_1, e_2)^2 \\ &= ad - b^2 = K. \end{aligned}$$

But R is determined by the metric, so the same is true of K.

For any tangent basis X, Y, the equivalent formula is

$$K = \frac{I(R(X, Y)Y, X)}{I(X, X)I(Y, Y) - I(X, Y)^2}.$$

The denominator is the squared area of the parallelogram spanned by X, Y.

We can now compute Gaussian curvature without choosing a normal. For

$$I = du^2 + r(u)^2 d\theta^2,$$

write $U = f_u$, $V = f_\theta$. The Christoffel symbols computed earlier say

$$\nabla_U U = 0, \quad \nabla_U V = \nabla_V U = \frac{r_u}{r} V, \quad \nabla_V V = -r r_u U.$$

Since $[U, V] = 0$ and $V(r) = 0$,

$$\begin{aligned} R(U,V)V &= \nabla_U(-r r_u U) - \nabla_V\left(\frac{r_u}{r} V\right) \\ &= -(r_u^2 + r r_{uu})U + r_u^2 U \\ &= -r r_{uu} U. \end{aligned}$$

The area denominator is r^2, so

$$K = -\frac{r_{uu}}{r}.$$

For a cylinder, $r = a$, giving $K = 0$. For a sphere, $r(u) = a\sin(u/a)$, giving $K = 1/a^2$.

A ***local isometry*** is a local diffeomorphism $\Phi : M \to \widetilde{M}$ satisfying

$$\widetilde{I}_{\Phi(p)}(d\Phi_p v, d\Phi_p w) = I_p(v, w).$$

It preserves the metric, so it preserves covariant differentiation, parallel transport, geodesics, and now Gaussian curvature:

$$\widetilde{K} \circ \Phi = K.$$

The plane and cylinder are locally isometric: both have metric $du^2 + dz^2$ in the coordinates used above. Their Gaussian curvatures are both zero, but their shape operators are different. With the outward cylinder normal,

$$L_{\text{plane}} = 0, \qquad [L_{\text{cylinder}}] = \begin{pmatrix} -1/a & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus mean curvature is not intrinsic: it is zero on the plane and $-1/(2a)$ on this oriented cylinder.

By contrast, no open piece of a sphere is isometric to an open piece of the plane: $1/a^2 \neq 0$. A sphere cannot be flattened without changing lengths, even locally.