# *Lecture 16: Curvature and Gauss–Bonnet*

For a graph $f(u,v) = (u, v, h(u,v))$, the upward unit normal and first fundamental form are

$$\nu = \frac{(-h_u, -h_v, 1)}{w}, \qquad g = \begin{pmatrix} 1 + h_u^2 & h_u h_v \\ h_u h_v & 1 + h_v^2 \end{pmatrix}, \quad w = \sqrt{1 + h_u^2 + h_v^2}.$$

Taking inner products of the second derivatives with $\nu$ gives $[II] = w^{-1}\,\mathrm{Hess}(h)$. Since $\det g = w^2$,

$$K = \frac{\det(II)}{\det g} = \frac{h_{uu}h_{vv} - h_{uv}^2}{(1 + h_u^2 + h_v^2)^2}.$$

For the paraboloid

$$h(u,v) = \frac{u^2 + v^2}{2},$$

the Hessian is the identity matrix. Therefore

$$K(u,v) = \frac{1}{(1 + u^2 + v^2)^2} > 0.$$

Curvature is largest at the vertex and tends to zero far from it. Unlike the sphere, this surface has positive but nonconstant Gaussian curvature.

As in Lecture 15, this also determines the full tensor:

$$R(X,Y)Z = K\big(I(Y,Z)X - I(X,Z)Y\big).$$

For $a > 0$, parametrize a helicoid by

$$f(u, v) = (u \cos v, u \sin v, av).$$

The coordinate tangent vectors satisfy

$$\begin{aligned} f_u &= (\cos v, \sin v, 0), \\ f_v &= (-u \sin v, u \cos v, a), \end{aligned}$$

so

$$I = du^2 + (u^2 + a^2)dv^2.$$

This has the geodesic parallel form from Lecture 15:

$$I = du^2 + q(u)^2 dv^2,$$

where

$$q = \sqrt{u^2 + a^2}, \qquad q_u = \frac{u}{\sqrt{u^2 + a^2}}, \qquad q_{uu} = \frac{a^2}{(u^2 + a^2)^{3/2}}.$$

Consequently,

$$K = -\frac{q_{uu}}{q} = -\frac{a^2}{(u^2 + a^2)^2} < 0.$$

The curvature is most negative along the axis $u = 0$, where $K = -1/a^2$, and tends to zero as $|u|$ grows.

The formula $K = -q_{uu}/q$ applies because of the *metric*, even though the helicoid is not a surface of revolution. No normal vector was needed in this calculation.

For a ring torus with major radius $a > b > 0$, use angles $\phi, \theta$ and write

$$f(\phi, \theta) = \big((a + b\cos\phi)\cos\theta, (a + b\cos\phi)\sin\theta, b\sin\phi\big).$$

The computation in Lecture 15 gives

$$I = b^2 d\phi^2 + (a + b\cos\phi)^2 d\theta^2, \qquad K = \frac{\cos\phi}{b(a + b\cos\phi)}.$$

The area element is

$$dA = b(a + b\cos\phi)\, d\phi\, d\theta.$$

Thus the positive and negative curvature contributions cancel:

$$\int_M K\, dA = \int_0^{2\pi} \int_0^{2\pi} \cos\phi\, d\phi\, d\theta = 0.$$

Integrating over a full period counts the surface once, apart from the identified edges, which have area zero.

For a sphere of radius $a$, by contrast,

$$\int_{S_a^2} K\, dA = \frac{1}{a^2} \cdot 4\pi a^2 = 4\pi.$$

Neither answer depends on the radii. Gauss–Bonnet will explain these integrals using the topology of the surface.

Fix an orientation of a surface by a unit normal $\nu$. For a unit tangent vector $T$, define its leftward rotation in the tangent plane by

$$JT = \nu \times T.$$

Let $c(s)$ be a unit-speed curve and $T = c'(s)$. Because $I(\nabla_T T, T) = 0$, there is a scalar ***signed geodesic curvature***

$$\kappa_g = I(\nabla_T T, JT) = \langle c'', JT \rangle, \qquad \nabla_T T = \kappa_g JT.$$

This is Kühnel's Definition 4.37. It measures turning within the surface. A unit-speed curve is geodesic exactly when $\kappa_g = 0$.

The full acceleration decomposes as

$$c'' = \kappa_g JT + \kappa_n \nu, \qquad \kappa_n = II(T, T), \qquad \kappa^2 = \kappa_g^2 + \kappa_n^2.$$

Here $\kappa = \|c''\|$ is the ordinary, nonnegative curvature of the space curve.

A ***positively oriented boundary*** has its region on the left: $JT$ points into the region. With $\nu$ fixed, reversing the direction of a curve reverses the sign of $\kappa_g$.

For the circle of radius $a$ in the upward-oriented plane,

$$c(s) = \big(a\cos(s/a), a\sin(s/a), 0\big),$$

put $t = s/a$. Then

$$T = (-\sin t, \cos t, 0), \qquad JT = (-\cos t, -\sin t, 0), \qquad c'' = \frac{1}{a}JT.$$

The circle, traversed counterclockwise, bounds a disk on its left. Hence

$$\kappa_g = \frac{1}{a}, \qquad \int_{\partial D} \kappa_g \, ds = \frac{1}{a} \cdot 2\pi a = 2\pi.$$

Now view the same curve as a horizontal circle on the cylinder $x^2 + y^2 = a^2$. Its outward normal is $\nu = (\cos t, \sin t, 0)$, and

$$c'' = -\frac{1}{a}\nu, \qquad \nabla_T T = 0, \qquad \kappa_g = 0.$$

Its ordinary curvature is still $1/a$, but its curvature within the cylinder vanishes. Thus geodesic curvature depends on the surface containing the curve.

The circle bounds a disk in the plane. It does not bound a disk contained in the cylinder. This distinction matters in applying Gauss–Bonnet.

On the unit sphere, fix a colatitude $0 < \alpha < \pi$. An arclength parametrization of the latitude is

$$c(s) = \big(\sin\alpha\cos(s/\sin\alpha), \sin\alpha\sin(s/\sin\alpha), \cos\alpha\big).$$

Put $t = s/\sin\alpha$. With the outward normal $\nu = c$,

$$\begin{aligned} T &= (-\sin t, \cos t, 0), \\ JT &= (-\cos\alpha\cos t, -\cos\alpha\sin t, \sin\alpha), \\ c'' &= \frac{1}{\sin\alpha}(-\cos t, -\sin t, 0). \end{aligned}$$

Therefore

$$\kappa_g = \langle c'', JT\rangle = \cot\alpha.$$

The cap containing the north pole lies on the left. Since the boundary has length $2\pi\sin\alpha$,

$$\int_{\partial D} \kappa_g \, ds = 2\pi\cos\alpha.$$

At the equator this integral is zero. For a cap larger than a hemisphere, $\alpha > \pi/2$, the boundary has negative geodesic curvature with this orientation.

Keeping the sign of $\kappa_g$ is essential. Its absolute value would lose the distinction between these caps.

**Gauss–Bonnet for a disk (Kühnel 4.38).** Let $D$ be a compact region in an oriented surface, diffeomorphic to a closed disk, with smooth boundary. For now, assume $D$ lies in a single coordinate patch. Orient $\partial D$ with $D$ on its left. Then

$$\int_D K \, dA + \int_{\partial D} \kappa_g \, ds = 2\pi.$$

The first integral records curvature throughout the region. The second records the turning of its boundary within the surface.

For a planar disk, $K = 0$, so the formula reduces to the turning-tangent theorem:

$$\int_{\partial D} \kappa_g \, ds = 2\pi.$$

For a hemisphere of the unit sphere, the boundary is a geodesic, so all of the contribution comes from the interior:

$$\int_D K \, dA = \text{Area}(D) = 2\pi.$$

The right-hand side depends on the disk topology, not on its size or shape. For a region with holes, it must change.

We will prove the formula using the turning-tangent theorem and the surface version of Green's theorem.

Let $D_\alpha$ be the cap on the unit sphere from the north pole down to colatitude $\alpha$, where $0 < \alpha < \pi$. In spherical coordinates,

$$I = d\phi^2 + \sin^2\phi\, dt^2, \qquad dA = \sin\phi\, d\phi\, dt.$$

The coordinates degenerate at the pole, but the integral is obtained by taking the limit as the lower bound on $\phi$ tends to zero.

Since $K = 1$,

$$\begin{aligned}\int_{D_\alpha} K\, dA &= \int_0^{2\pi}\int_0^{\alpha} \sin\phi\, d\phi\, dt \\ &= 2\pi(1 - \cos\alpha).\end{aligned}$$

The latitude computation gives

$$\int_{\partial D_\alpha} \kappa_g\, ds = 2\pi\cos\alpha.$$

Adding proves the formula directly:

$$2\pi(1 - \cos\alpha) + 2\pi\cos\alpha = 2\pi.$$

As the cap grows, the integrated interior curvature increases and the boundary contribution decreases. For caps larger than a hemisphere, the boundary term is negative.

A ***one-form*** $\eta$ assigns a linear functional $\eta_p : T_pM \to \mathbb{R}$ to each point, varying smoothly with $p$ (Kühnel 4.33).

In coordinates $(u, v)$, write

$$\eta = P\,du + Q\,dv.$$

This means that for $X = x^1 f_u + x^2 f_v$,

$$\eta(X) = Px^1 + Qx^2.$$

The symbols $du, dv$ are the linear functionals that read off the two coordinate components of a tangent vector.

For a curve $c(t) = f(u(t), v(t))$, the line integral is

$$\int_c \eta = \int_a^b \eta(\dot{c}(t))\,dt = \int_a^b \left(P\dot{u} + Q\dot{v}\right) dt.$$

Thus one-form notation packages the line integrals from multivariable calculus. The integral is independent of an orientation-preserving change of parameter.

For a scalar function $h$, its differential is the example

$$dh = h_u\,du + h_v\,dv, \qquad dh(X) = Xh.$$

The ***exterior derivative*** of $\eta = P\,du + Q\,dv$ is

$$d\eta = (Q_u - P_v)\,du \wedge dv.$$

The two-form $du \wedge dv$ evaluates on two vectors by the determinant of their coordinate components:

$$(du \wedge dv)(X, Y) = x^1 y^2 - x^2 y^1.$$

An equivalent coordinate-independent formula is

$$d\eta(X, Y) = X\big(\eta(Y)\big) - Y\big(\eta(X)\big) - \eta([X, Y]).$$

The bracket term makes this expression pointwise and bilinear in $X, Y$.

**Stokes's theorem (Kühnel 4.36).** If $\eta$ is defined on a neighborhood of a compact oriented region $D$, then

$$\int_D d\eta = \int_{\partial D} \eta.$$

In coordinates this is precisely Green's theorem:

$$\iint (Q_u - P_v)\,du\,dv = \oint P\,du + Q\,dv.$$

In positively oriented coordinates, the area form is

$$dA = \sqrt{\det g}\,du \wedge dv.$$

Choose a smooth, positively oriented orthonormal tangent frame $E_1, E_2$ on a neighborhood of the entire disk $D$. Such a frame comes from applying Gram–Schmidt to the coordinate tangent fields, with $E_2 = \nu \times E_1$.

Define the ***connection one-form***

$$\eta(X) = I(\nabla_X E_2, E_1).$$

Metric compatibility gives

$$\nabla_X E_1 = -\eta(X) E_2, \qquad \nabla_X E_2 = \eta(X) E_1.$$

Indeed, differentiating $I(E_i, E_i) = 1$ removes the component along $E_i$, and differentiating $I(E_1, E_2) = 0$ gives the opposite coefficients.

The form $\eta$ records the rotation of the frame under covariant differentiation. It contains the same connection information as Christoffel symbols, in an orthonormal frame.

This is Kühnel's form $\omega_2^1$; its exterior derivative will recover Gaussian curvature.

It is important that the frame extend across *all* of $D$. A polar frame singular at the center of a disk does not satisfy this requirement.

Using the frame equations, compute

$$\begin{aligned}\nabla_X \nabla_Y E_2 &= \nabla_X \big(\eta(Y) E_1\big) \\ &= X\big(\eta(Y)\big) E_1 - \eta(Y)\eta(X) E_2.\end{aligned}$$

Swap $X, Y$ and subtract; the products cancel. The bracket term is

$$\nabla_{[X,Y]} E_2 = \eta([X, Y]) E_1.$$

Hence

$$R(X, Y) E_2 = d\eta(X, Y) E_1.$$

In particular, the formula for Gaussian curvature gives

$$d\eta(E_1, E_2) = I(R(E_1, E_2) E_2, E_1) = K.$$

Every alternating bilinear form on a two-dimensional vector space is determined by its value on a basis. Because $dA(E_1, E_2) = 1$, we obtain

$$d\eta = K\, dA.$$

This is the surface case of Kühnel 4.35. It expresses curvature as a derivative of the connection form.

Stokes's theorem therefore gives

$$\int_D K\, dA = \int_{\partial D} \eta.$$

The next step relates this boundary integral to the turning of the boundary tangent.

Parametrize $\partial D$ positively by arclength $c : [0, \ell] \to M$. Choose a continuous real angle $\theta(s)$ along the interval such that

$$T = c' = \cos\theta\, E_1 + \sin\theta\, E_2.$$

The angle need not have the same value at the two endpoints, even though $T(\ell) = T(0)$. Its leftward normal is

$$JT = -\sin\theta\, E_1 + \cos\theta\, E_2.$$

Differentiate using the connection equations:

$$\nabla_T T = \big(\theta'(s) - \eta(T)\big) JT.$$

Thus

$$\kappa_g = \theta' - \eta(T).$$

The angle derivative alone omits the rotation of the chosen frame. The connection term corrects for it.

Since the frame extends across the disk, the turning-tangent theorem gives

$$\theta(\ell) - \theta(0) = 2\pi.$$

One justification is to deform the coordinate metric continuously to the Euclidean metric, applying Gram–Schmidt throughout. The winding number stays integer-valued and constant, and in the Euclidean case it is 1.

*Proof of Gauss–Bonnet for a disk.* Integrate the identity from the preceding page:

$$\begin{aligned}\int_{\partial D} \kappa_g \, ds &= \int_0^{\ell} \theta'(s) \, ds - \int_{\partial D} \eta \\ &= 2\pi - \int_{\partial D} \eta \\ &= 2\pi - \int_D d\eta \\ &= 2\pi - \int_D K \, dA.\end{aligned}$$

The second line is the turning-tangent theorem, the third is Stokes's theorem, and the last uses the curvature equation. Rearranging gives

$$\int_D K \, dA + \int_{\partial D} \kappa_g \, ds = 2\pi.$$

Although we introduced a frame to prove it, the final identity uses only intrinsic quantities.

The disk hypothesis entered in choosing a frame that extends over the region and obtaining one full turn of the positively oriented boundary tangent. It cannot be discarded while keeping the right-hand side equal to $2\pi$.

The second fundamental form did not enter this proof. The same reasoning applies to an oriented surface equipped with an intrinsic Riemannian metric.

Suppose the boundary of a disk is only piecewise smooth. At a corner let $\beta_i$ be the interior angle, and set

$$\varepsilon_i = \pi - \beta_i, \qquad 0 < \beta_i < 2\pi.$$

The ***signed exterior angle*** $\varepsilon_i$ is the jump in tangent direction. A convex corner gives a positive turn, while a reentrant corner gives a negative turn.

**Gauss–Bonnet with corners (Kühnel 4.39).**

$$\int_D K \, dA + \int_{\partial D} \kappa_g \, ds + \sum_i \varepsilon_i = 2\pi.$$

The boundary integral is the sum over the smooth edges.

*Proof idea.* Add the smooth turning along each edge and the jumps at the corners:

$$\sum_{\text{edges}} \int \theta' \, ds + \sum_i \varepsilon_i = 2\pi.$$

The same connection-form calculation proves the formula. Equivalently, round the corners and take a limit.

For a planar polygon, $K = 0$ and each straight edge has $\kappa_g = 0$. The formula reduces to the familiar identity $\sum_i \varepsilon_i = 2\pi$.

A ***geodesic triangle*** bounds a disk $D$ with three geodesic edges. Let its interior angles be $\beta_1, \beta_2, \beta_3$.

Since $\kappa_g = 0$ on every edge, Gauss–Bonnet gives

$$\int_D K \, dA + \sum_{i=1}^{3} (\pi - \beta_i) = 2\pi.$$

Thus

$$\beta_1 + \beta_2 + \beta_3 - \pi = \int_D K \, dA.$$

The ***angle excess*** is the integrated curvature (Kühnel 4.40–4.41).

If $K$ is everywhere positive, zero, or negative on the interior, the angle sum is respectively greater than, equal to, or less than $\pi$.

For constant curvature $K_0$,

$$\beta_1 + \beta_2 + \beta_3 - \pi = K_0 \operatorname{Area}(D).$$

In particular, on a sphere of radius $a$,

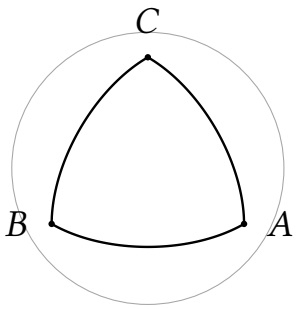
$$\operatorname{Area}(D) = a^2(\beta_1 + \beta_2 + \beta_3 - \pi).$$

The enclosed disk must belong to the surface. A closed chain of three geodesic segments alone is not enough.

On the sphere of radius $a$, consider the octant

$$D = \{(x, y, z) \in S_a^2 : x, y, z \geq 0\}.$$

Its vertices are $A = (a, 0, 0)$, $B = (0, a, 0)$, and $C = (0, 0, a)$. The edges are quarter-arcs of great circles.




At each vertex the two edge tangents are perpendicular in the tangent plane, so all three interior angles are $\pi/2$. The angle formula predicts

$$\text{Area}(D) = a^2 \left( \frac{3\pi}{2} - \pi \right) = \frac{\pi a^2}{2}.$$

This agrees with one eighth of the sphere's area. The three right angles are intrinsic angles, even though the planar drawing distorts them.

The same connection form relates curvature to parallel transport. Let $V$ be a parallel unit vector field along the positively oriented boundary of a disk. Write

$$V(s) = \cos\varphi(s)E_1 + \sin\varphi(s)E_2.$$

Differentiating as before gives

$$\nabla_T V = (\varphi' - \eta(T))JV.$$

Thus parallel transport satisfies $\varphi' = \eta(T)$, and

$$\varphi(\ell) - \varphi(0) = \int_{\partial D} \eta = \int_D K\, dA.$$

The endpoint frame is the same as the initial frame, so transport around the loop is rotation through angle

$$\int_D K\, dA \pmod{2\pi}.$$

This rotation is called the ***holonomy*** of the loop.

For a unit-sphere cap it is

$$2\pi(1 - \cos\alpha) \equiv -2\pi\cos\alpha \pmod{2\pi}.$$

This agrees with solving the parallel-transport equations around a latitude. The hypothesis that the loop bounds a disk is important here as well.

Decompose a compact surface into finitely many vertices, edges, and disk-shaped faces. Its ***Euler characteristic*** is

$$\chi(M) = V - E + F.$$

Here $V, E, F$ count the vertices, edges, and faces, with each shared edge counted only once. A triangulation is a convenient example of such a decomposition.

This integer is independent of the decomposition and is a topological invariant. Basic subdivisions preserve it: inserting a vertex in an edge increases both $V$ and $E$ by one, while splitting a face by an edge increases both $E$ and $F$ by one.

A sphere has a tetrahedral decomposition, so

$$\chi(S^2) = 4 - 6 + 4 = 2.$$

A torus is a square with opposite sides identified. Its four corners become one vertex, and the sides give two edges, with one face. Hence

$$\chi(T^2) = 1 - 2 + 1 = 0.$$

The square description is a cell decomposition rather than a triangulation, but computes the same invariant.

The global Gauss–Bonnet theorem identifies total curvature with $2\pi$ times this integer.

**Global Gauss–Bonnet (Kühnel 4.43).** For a compact oriented surface $M$ without boundary,

$$\int_M K \, dA = 2\pi \chi(M).$$

*Proof sketch.* Choose a finite smooth triangulation fine enough that each face lies in a coordinate patch. The edges need not be geodesics. For each triangular face $D_j$, the corner formula says

$$\int_{D_j} K \, dA + \int_{\partial D_j} \kappa_g \, ds = \sum_{\text{corners of } D_j} \beta - \pi.$$

Sum over all faces. Each shared edge is traversed in opposite directions by its two faces, so the boundary integrals cancel.

The interior angles meeting at each vertex sum to $2\pi$. Consequently,

$$\int_M K \, dA = 2\pi V - \pi F.$$

Every triangular face has three edges and each edge belongs to two faces, so $3F = 2E$. Therefore

$$2\pi V - \pi F = 2\pi(V - E + F) = 2\pi \chi(M).$$

For a connected compact orientable surface without boundary, the classification of surfaces gives

$$\chi(M) = 2 - 2g,$$

where the ***genus*** $g$ is the number of handles. Gauss–Bonnet becomes

$$\int_M K \, dA = 4\pi(1 - g).$$

For genus zero, this is $4\pi$. Thus an ellipsoid has the same total curvature as a round sphere, even though its Gaussian curvature is generally not constant.

For genus one, the total is zero. This explains the cancellation in the torus calculation. If curvature is positive somewhere on such a surface, it must also be negative somewhere.

For genus at least two, the total curvature is negative. This forces negative curvature somewhere, not necessarily at every point.

In particular, if $K > 0$ everywhere on a compact connected orientable surface without boundary, then

$$\chi(M) > 0, \qquad g = 0.$$

Changing the geometry without changing the topology can redistribute Gaussian curvature, but cannot change its integral.

For a compact oriented surface $D$ with piecewise smooth boundary, the general formula is

$$\int_D K\,dA + \int_{\partial D} \kappa_g\,ds + \sum_i \varepsilon_i = 2\pi\chi(D).$$

Each boundary component is oriented with the region on its left. The same subdivision proof cancels interior edges and retains the actual boundary terms.

For a planar annulus $r \le \sqrt{x^2+y^2} \le R$, where $0 < r < R$, the outer circle is counterclockwise and the inner circle is clockwise. Thus

$$\int_{\text{outer}} \kappa_g\,ds = 2\pi, \qquad \int_{\text{inner}} \kappa_g\,ds = -2\pi.$$

Since $K = 0$ and there are no corners, the left-hand side is zero, as required by $\chi(\text{annulus}) = 0$.

More generally, a connected orientable surface of genus $g$ with $b$ boundary components has

$$\chi(D) = 2 - 2g - b.$$

For a disk this is $1$, and for an annulus it is $0$.

If a smooth deformation preserves the geometry near the boundary and preserves the topology, Gauss–Bonnet forces the integrated Gaussian curvature to stay unchanged.