MA 563: Problem sheet 3 (on Kühnel chapter 4)

Notation and conventions. All maps and fields are smooth. Work locally on an embedded hypersurface patch $f : U \subset \mathbb{R}^n \to M \subset \mathbb{R}^{n+1}$, with local unit normal ν. Write $f_i = \partial_i f$, $f_{ij} = \partial_i \partial_j f$, and $I(X, Y) = \langle X, Y \rangle$. For $X = \sum_i \xi^i f_i$, scalar functions φ, and fields Y along f, set

$$X(\varphi) = \sum_{i=1}^{n} \xi^i \partial_i \varphi, \qquad D_X Y = \sum_{i=1}^{n} \xi^i \partial_i Y, \qquad \nabla_X Y = (D_X Y)^{\top}.$$

Here $^{\top}$ denotes orthogonal projection onto the tangent space; functions and fields are expressed in the coordinates of f. Use $L = -D\nu \circ (Df)^{-1}$, as in class. Dots denote an arbitrary curve parameter t; primes on curves denote arclength s. In the problems, X, Y, Z are tangent vector fields unless otherwise stated. Give proofs and justify your computations.

Problem 1. The Lie bracket and the Jacobi identity

For tangent vector fields X, Y, define their *Lie bracket* (or *Jacobi bracket*) by

$$[X, Y] = D_X Y - D_Y X.$$

(a) Write $X = \sum_i \xi^i f_i$ and $Y = \sum_j \eta^j f_j$. Prove that the bracket is tangent and that

$$[X, Y] = \sum_{j=1}^{n} \big(X(\eta^j) - Y(\xi^j) \big) f_j.$$

Explain why the terms involving f_{ij} cancel.

(b) Show that $[X, Y](\varphi) = X(Y(\varphi)) - Y(X(\varphi))$. Use this characterization to prove the Jacobi identity

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

Hint: Expand the commutators as compositions of operators on functions. Why does equality on all functions imply equality of fields?

(c) For smooth scalar functions a, b, prove

$$[aX, bY] = ab[X, Y] + aX(b)Y - bY(a)X.$$

Explain why the bracket is not bilinear over smooth functions.

Problem 2. Coordinate fields commute

Regard f_i as tangent vector fields on the image of the patch.

(a) Show that $D_{f_i} f_j = f_{ij}$, and deduce both

$$\nabla_{f_i} f_j = \nabla_{f_j} f_i, \qquad [f_i, f_j] = 0.$$

Indicate exactly where equality of mixed partial derivatives is used.

(b) Using the tangency established in Problem 1, prove

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

for arbitrary tangent fields. This is the *torsion-free identity*. Does it say that $\nabla_X Y = \nabla_Y X$ for all X, Y?

Problem 3. The second fundamental form

Define the *vector-valued second fundamental form* by

$$B(X, Y) = D_X Y - \nabla_X Y.$$

(a) Prove that $B(X, Y)$ is normal, symmetric, and bilinear over smooth functions:

$$B(aX, bY) = abB(X, Y).$$

Conclude that $B(X, Y)_p$ depends only on X_p, Y_p, not on the chosen extensions of these vectors to fields. *Hint:* The derivative term in $D_X(bY)$ is tangent.

(b) Set $II(X, Y) = \langle B(X, Y), v \rangle$. Prove the *Gauss formula* and the relation to the shape operator:

$$D_X Y = \nabla_X Y + II(X, Y)v, \qquad II(X, Y) = -\langle Y, D_X v \rangle = I(LX, Y).$$

Thus, for a surface in $\mathbb{R}^3$, this agrees with our previous definition of the second fundamental form. In particular, verify

$$II(f_i, f_j) = \langle f_{ij}, v \rangle = h_{ij}.$$

(c) What happens to L, II, and B when v is replaced by $-v$? Explain why the vector-valued discrepancy is independent of this choice.

Problem 4. An orthonormal frame need not be a coordinate frame

On a polar-coordinate patch in the plane, use

$$f(r,\theta) = (r\cos\theta, r\sin\theta, 0), \qquad r > 0,$$

with θ restricted to an open interval of length less than 2π. Put $e_r = f_r$ and $e_\theta = r^{-1} f_\theta$.

(a) Verify that (e_r, e_θ) is orthonormal. Compute

$$\nabla_{e_r} e_r, \quad \nabla_{e_r} e_\theta, \quad \nabla_{e_\theta} e_r, \quad \nabla_{e_\theta} e_\theta.$$

Use the torsion-free identity to compute $[e_r, e_\theta]$, and check your answer directly using Problem 1(c).

(b) Explain why $[f_r, f_\theta] = 0$ but $[e_r, e_\theta] \neq 0$. Can e_r, e_θ be the two coordinate fields of a single coordinate system on this patch? Why does their nonzero bracket not imply that the plane has nonzero Gaussian curvature?

Problem 5. The first fundamental form determines the connection

Let $g_{ij} = I(f_i, f_j)$, and let (g^{ij}) denote the inverse matrix.

(a) Prove *metric compatibility* directly from orthogonal projection:

$$X\big(I(Y,Z)\big) = I(\nabla_X Y, Z) + I(Y, \nabla_X Z).$$

Apply this identity to coordinate fields, together with Problem 2, to show

$$2I(\nabla_{f_i} f_j, f_k) = \partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}.$$

(b) Define the Christoffel symbols by $\nabla_{f_i} f_j = \sum_{k=1}^{n} \Gamma_{ij}^k f_k$. Deduce

$$\Gamma_{ij}^k = \frac{1}{2} \sum_{\ell=1}^{n} g^{k\ell} \big(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij}\big).$$

Explain why these coefficients, and hence covariant differentiation, depend only on I, even though the definition initially used the ambient Euclidean space. What happens when all g_{ij} are constant?

Problem 6. A plane and a cylinder

For $R > 0$, compare the parametrizations

$$P(u, v) = (u, v, 0), \qquad F(u, v) = \big(R\cos(u/R), R\sin(u/R), v\big).$$

Work on an open rectangle whose u-interval has length less than $2\pi R$. Use the upward normal for the plane and the outward normal for the cylinder.

(a) Compute the two first fundamental forms. Prove that $P(u, v) \mapsto F(u, v)$ is a local isometry, meaning a local diffeomorphism preserving inner products of tangent vectors.

(b) Compute all four coordinate covariant derivatives for each parametrization, first directly by projection and then using Problem 5. Compute the matrices of II and L for both surfaces.

(c) Identify a nonzero ordinary derivative $D_{F_i}F_j$ whose tangential projection vanishes. Explain what geometric information is discarded by covariant differentiation. Does equality of the two first fundamental forms force equality of their second fundamental forms?

Problem 7. Geodesics

For a field $V(t)$ along a curve $c(t)$ on a surface, write $\nabla_{\dot c} V = (\dot V)^{\top}$. A parametrized curve is a *geodesic* if $\nabla_{\dot c}\dot c = 0$.

(a) Prove the acceleration decomposition

$$\ddot c = \nabla_{\dot c}\dot c + II(\dot c, \dot c)\nu.$$

Show that a geodesic has constant speed. For a unit-speed curve, deduce

$$\kappa^2 = \|\nabla_{c'} c'\|^2 + II(c', c')^2, \qquad \kappa = \|c''\|.$$

In particular, explain why being a geodesic does not mean $\kappa = 0$.

(b) For the cylinder in Problem 6, let $c(s) = F(as, bs)$, where a, b are constants satisfying $a^2+b^2 = 1$. Show that this curve is unit speed and geodesic, and compute its ordinary curvature κ. Describe the cases $a = 0$, $b = 0$, and $ab \neq 0$. The formula for F may here be used for all real u, v.

Problem 8. Vector fields on a sphere

Let $R > 0$ and $S_R^2 = \{p \in \mathbb{R}^3 : \|p\| = R\}$, with outward unit normal $\nu(p) = p/R$. For a fixed vector $a \in \mathbb{R}^3$, define

$$X_a(p) = a - \frac{\langle a, p\rangle}{R^2} p.$$

(a) Show that X_a is a smooth tangent vector field. For an arbitrary tangent field X, differentiate in the ambient space and then project to prove

$$\nabla_X X_a = -\frac{\langle a, p\rangle}{R^2} X.$$

Explain why projecting a constant ambient vector does not generally produce a covariantly constant tangent field.

(b) Prove that

$$[X_a, X_b](p) = \frac{1}{R^2}\big(\langle a, p\rangle X_b(p) - \langle b, p\rangle X_a(p)\big).$$

Take $a = (1, 0, 0)$, $b = (0, 1, 0)$, and $p = (0, 0, R)$. Show that the bracket vanishes at this point but not throughout any neighborhood of it. Why is vanishing at a single point insufficient to conclude that the fields are coordinate fields near that point?

(c) Compute L and II on the outward-oriented sphere, checking the signs carefully. Explain why the vanishing bracket at the north pole does not imply zero Gaussian curvature there.

Problem 9. Parallel transport in a moving frame on the plane

A tangent field $V(t)$ along a curve $c(t)$ on a surface is *parallel* if $\nabla_{\dot{c}} V = 0$. Parallel transport takes an initial vector $V(0)$ to the value $V(t)$ of this parallel field. Consider the circle in the plane $M = \{z = 0\}$,

$$c(t) = (\rho \cos t, \rho \sin t, 0), \qquad \rho > 0, \quad 0 \le t \le 2\pi.$$

The vectors being transported are tangent to the *plane*, not necessarily tangent to the circle. Along c, use the orthonormal frame

$$e_r(t) = (\cos t, \sin t, 0), \qquad e_\theta(t) = (-\sin t, \cos t, 0),$$

and write $V(t) = a(t)e_r(t) + b(t)e_\theta(t)$. This frame is defined along the whole curve, without choosing a single polar-coordinate chart covering the full circle.

(a) Differentiate the frame and show that the parallel-transport equation is

$$\dot{a} - b = 0, \qquad \dot{b} + a = 0.$$

Why does covariant differentiation along this curve agree with ordinary ambient differentiation of V?

(b) Solve this system for arbitrary initial values $a(0) = a_0$, $b(0) = b_0$. Express the answer both in the moving frame and in the fixed Cartesian frame of the plane.

(c) Show directly from the differential equations that $a(t)^2 + b(t)^2$ is constant. Determine the parallel-transport map $T_{c(0)}M \to T_{c(0)}M$ after one full circuit. Explain how nonconstant coefficients in the moving frame can describe a constant ambient vector.

(d) Is the velocity field $\dot{c}(t) = \rho e_\theta(t)$ parallel along the circle? Relate your answer to the geodesic equation.

Problem 10. Parallel transport around a latitude

On the unit sphere, fix a colatitude $0 < \alpha < \pi$ and consider

$$c(t) = (\sin\alpha\cos t, \sin\alpha\sin t, \cos\alpha), \qquad 0 \le t \le 2\pi.$$

Use the tangent frame

$$E_1(t) = (\cos\alpha\cos t, \cos\alpha\sin t, -\sin\alpha), \qquad E_2(t) = (-\sin t, \cos t, 0).$$

The outward unit normal along the curve is $\nu(t) = c(t)$.

(a) Verify that E_1, E_2 form an orthonormal tangent frame. Differentiate in $\mathbb{R}^3$ and project onto the tangent plane to prove

$$\nabla_{\dot c} E_1 = \cos\alpha\, E_2, \qquad \nabla_{\dot c} E_2 = -\cos\alpha\, E_1.$$

(b) Write a parallel field as $V = aE_1 + bE_2$. Derive its differential equations and solve them with $a(0) = a_0$, $b(0) = b_0$. Give an explicit formula for $V(t)$. *Hint:* The coefficient system has the same form as in Problem 9, with a constant factor $\cos\alpha$.

(c) Since the frame at $t = 2\pi$ agrees with the frame at $t = 0$, the parallel-transport map around the loop is a linear map of one tangent plane to itself. Compute its matrix and show that it is a rotation. Specify the rotation angle modulo 2π, taking positive angles from $E_1(0)$ toward $E_2(0)$. What happens for $\alpha = \pi/2$ and $\alpha = \pi/3$?

(d) For $\alpha = \pi/3$, can a nonzero parallel field satisfy $V(2\pi) = V(0)$? Explain why your answer does not contradict existence and uniqueness of parallel transport for every initial vector on the interval $[0, 2\pi]$. Compare with Problem 9 and explain why preserving lengths and angles does not force transport around a loop to be the identity.

Problem 11. Identities of the curvature tensor

Use the convention from lecture

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Here X, Y, Z, W are tangent fields on an embedded hypersurface, and a, b, c below are smooth scalar functions. You may use the Gauss equation

$$R(X,Y)Z = II(Y,Z)LX - II(X,Z)LY, \qquad II(X,Y) = I(LX,Y),$$

in parts (b)–(d).

(a) Starting from the definition of R, prove

$$R(Y,X)Z = -R(X,Y)Z, \qquad R(aX,bY)(cZ) = abc\,R(X,Y)Z.$$

Identify the cancellation of derivatives of the coefficient functions. Conclude that $R(X,Y)Z$ at a point depends only on the three tangent vectors at that point.

(b) Prove the following identities, including all signs:

$$I(R(X,Y)Z,W) = -I(R(X,Y)W,Z),$$
$$I(R(X,Y)Z,W) = I(R(Z,W)X,Y).$$

The first says that $R(X,Y)$ is skew-adjoint with respect to I. *Hint:* Pair the Gauss equation with W and use the symmetry of II.

(c) Prove the *first Bianchi identity*

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0.$$

As an optional second proof, use only the torsion-free identity and the Jacobi identity from Problems 1–2, without the Gauss equation.

(d) Now assume the hypersurface is two-dimensional. At a point, choose orthonormal principal directions e_1, e_2, with $Le_i = \kappa_i e_i$. Use the Gauss equation to establish

$$R(e_1,e_2)e_2 = Ke_1, \qquad R(e_1,e_2)e_1 = -Ke_2, \qquad K = \kappa_1\kappa_2.$$

Explain why the mixed values of II vanish in this basis. Deduce, for arbitrary tangent vectors,

$$R(X,Y)Z = K\big(I(Y,Z)X - I(X,Z)Y\big).$$

What is the matrix of $R(e_1,e_2)$, with vectors represented as columns? Why does $K = 0$ at a point imply that the whole curvature tensor vanishes there for a surface?

Problem 12. Isometry versus diffeomorphism

Let $M, \widetilde{M}$ be surfaces with first fundamental forms $I, \widetilde{I}$. A diffeomorphism $\Phi : M \to \widetilde{M}$ is an *isometry* if

$$\widetilde{I}_{\Phi(p)}(d\Phi_p v, d\Phi_p w) = I_p(v, w) \qquad (v, w \in T_pM).$$

For a tangent field X, its pushforward is the field $\widetilde{X}_{\Phi(p)} = d\Phi_p(X_p)$.

(a) Suppose Φ is an isometry. Prove that it preserves the covariant derivative:

$$d\Phi(\nabla_X Y) = \widetilde{\nabla}_{\widetilde{X}} \widetilde{Y}.$$

You may use uniqueness of the torsion-free, metric-compatible connection. *Hint:* Pull back $\widetilde{\nabla}$ by $d\Phi$, and check these two properties. First show that $d\Phi([X, Y]) = [\widetilde{X}, \widetilde{Y}]$ by testing on functions.

(b) Deduce that an isometry preserves the curvature tensor:

$$d\Phi\big(R(X, Y)Z\big) = \widetilde{R}(\widetilde{X}, \widetilde{Y})\widetilde{Z}.$$

Use the formula for Gaussian curvature in an orthonormal basis to conclude that $\widetilde{K}(\Phi(p)) = K(p)$.

(c) Let $N = (0, 0, 1)$ be the north pole of the unit sphere. Show that

$$F(u, v) = \frac{(2u, 2v, u^2 + v^2 - 1)}{1 + u^2 + v^2}$$

defines a diffeomorphism $\mathbb{R}^2 \to S^2 \setminus \{N\}$, by writing and checking its smooth inverse. Compute the pullback of the spherical metric and show that

$$F^* I_{S^2} = \frac{4}{(1 + u^2 + v^2)^2}(du^2 + dv^2).$$

Conclude that this particular diffeomorphism is conformal but is not an isometry from the Euclidean plane.

(d) Prove that *no* isometry exists between $S^2 \setminus \{N\}$, with its induced spherical metric, and $\mathbb{R}^2$, with its Euclidean metric. Can even nonempty open subsets of these two surfaces be isometric? Explain why this does not contradict part (c), and specify the metric on $\mathbb{R}^2$ that *does* make F an isometry.