

Section 3: Selected exercise solutions

October 6, 2026

Problem 1. Verify that the matrix g_{ij} of the first fundamental form of $f : U \rightarrow \mathbb{R}^{n+1}$ can be written as a matrix product $(Df)^T \cdot (Df)$.

Solution. Write $U \subset \mathbb{R}^n$, with coordinates $u_1, \dots, u_n$, and $f = (f^1, \dots, f^{n+1})$. The columns of Df are the coordinate tangent vectors:

$$Df = \begin{pmatrix} | & & | \\ f_{u_1} & \cdots & f_{u_n} \\ | & & | \end{pmatrix}.$$

Consequently, the (i, j) -entry of the product is

$$\begin{aligned} ((Df)^T Df)_{ij} &= \sum_{\alpha=1}^{n+1} \frac{\partial f^\alpha}{\partial u_i} \frac{\partial f^\alpha}{\partial u_j} \\ &= \langle f_{u_i}, f_{u_j} \rangle = g_{ij}. \end{aligned}$$

Thus $\boxed{g = (Df)^T Df}$. Equivalently, for coordinate vectors $a, b \in \mathbb{R}^n$,

$$I(a, b) = \langle Df(a), Df(b) \rangle = a^T g b.$$

In particular, $a^T g a = \|Df(a)\|^2 > 0$ for $a \neq 0$, because f is an immersion.

Problem 2. Show that for a curve c inside a given surface element, the two following statements are equivalent:

- (i) c is a line of curvature.
- (ii) The ruled surface f defined by the surface normal v along c is developable (i.e., satisfies the equation $K = 0$). More precisely we have $f(u, v) = c(u) + v\nu(c(u))$.

Solution. Reparametrize the regular curve by arclength s , and put

$$T = c', \quad n(s) = \nu(c(s)), \quad X(s, v) = c(s) + vn(s).$$

We use the shape operator $S = -d\nu$, so $n' = -S(T)$. A line of curvature is a curve whose tangent is a principal direction: $S(T) = \lambda T$ for some function λ .

The derivatives of the ruled surface are

$$X_s = T + vn', \quad X_v = n, \quad X_{sv} = n', \quad X_{vv} = 0.$$

Since $T \perp n$ and $n' \perp n$, its first fundamental form has coefficients

$$E = \|T + vn'\|^2, \quad F = 0, \quad G = 1.$$

In particular, X is regular near $v = 0$.

Problem 2, continued.

At any regular point set

$$W = \|X_s \times X_v\|, \quad \eta = \frac{(T + vn') \times n}{W}.$$

The mixed coefficient of the second fundamental form is

$$b_{12} = \langle X_{sv}, \eta \rangle = \frac{\det(T, n, n')}{W},$$

because the term $\det(n', n, n')$ vanishes. Also $b_{22} = 0$ and $EG - F^2 = W^2$. Hence

$$K_X = -\frac{\det(T, n, n')^2}{W^4}.$$

Thus $K_X = 0$ if and only if T, n, n' are linearly dependent. But $T, n' \in n^\perp$, while $n \notin n^\perp$. This is equivalent to n' being a multiple of T , or to

$$S(T) = \lambda T.$$

This proves both implications.

The assertion concerns the regular part of the ruled surface. Indeed, when $n' = -\lambda T$, we have $X_s = (1 - v\lambda)T$; singular points can occur when $1 - v\lambda = 0$, but $K_X = 0$ wherever X is regular.

Problem 3. Let c be a curve parametrized by arc length, and suppose that its image is contained in a surface element $f : U \rightarrow \mathbb{R}^3$. The Darboux three-frame E_1, E_2, E_3 is then defined by the relations

$$E_1(s) = c'(s), \quad E_3(s) = \nu(c(s)), \quad E_2(s) = E_3(s) \times E_1(s).$$

Here, as usual, ν denotes the unit normal on the surface f . Derive the following derivative equations for this three-frame, which correspond to the Frenet equations:

$$\begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}' = \begin{pmatrix} 0 & \kappa_g & \kappa_\nu \\ -\kappa_g & 0 & \tau_g \\ -\kappa_\nu & -\tau_g & 0 \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}.$$

The notations are as follows. The geodesic curvature is $\kappa_g = \langle c'', E_2 \rangle$, the normal curvature is $\kappa_\nu = \Pi(c', c')$, and τ_g denotes a certain geodesic torsion.

Solution. The three vectors form an orthonormal frame. Differentiating $\langle E_i, E_j \rangle = \delta_{ij}$ gives

$$\langle E'_i, E_j \rangle = -\langle E_i, E'_j \rangle.$$

Thus the coefficient matrix must be skew-symmetric. It remains to identify its three independent entries.

Problem 3, continued.

Since $E_1 = c'$, its E_2 -coefficient is $\langle E'_1, E_2 \rangle = \kappa_g$. With $S = -d\nu$,

$$E'_3 = -S(E_1).$$

Differentiating $\langle E_1, E_3 \rangle = 0$, we obtain

$$\begin{aligned} \langle E'_1, E_3 \rangle &= -\langle E_1, E'_3 \rangle \\ &= \langle S(E_1), E_1 \rangle = \Pi(c', c') = \kappa_v. \end{aligned}$$

Define the geodesic torsion by

$$\tau_g = \langle S(E_1), E_2 \rangle = -\langle E'_3, E_2 \rangle.$$

Skew-symmetry now gives all three equations:

$\begin{aligned} E'_1 &= \kappa_g E_2 + \kappa_v E_3, \\ E'_2 &= -\kappa_g E_1 + \tau_g E_3, \\ E'_3 &= -\kappa_v E_1 - \tau_g E_2. \end{aligned}$
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In particular, $\tau_g = 0$ precisely when E_1 is a principal direction. The geodesic torsion is generally different from the ordinary Frenet torsion τ ; Problem 8 explains why they agree along an asymptotic Frenet curve with these conventions.

Problem 5. *A surface of rotation can always be locally parametrized in such a way that the new parametrization is angle preserving. Hint: find $\psi = \psi(u)$ with the property that $(u, v) \mapsto f(\psi(u), v)$ is angle preserving.*

Solution. On a regular rotational patch write

$$f(t, v) = (r(t) \cos v, r(t) \sin v, h(t)), \quad r(t) > 0.$$

Put $q(t) = \sqrt{\dot{r}(t)^2 + \dot{h}(t)^2} > 0$. Direct differentiation gives

$$I = q(t)^2 dt^2 + r(t)^2 dv^2.$$

Define a new coordinate by

$$u(t) = \int_{t_0}^t \frac{q(\xi)}{r(\xi)} d\xi.$$

Because $du/dt = q/r > 0$, this has a local inverse $t = \psi(u)$, with $d\psi/du = r(\psi)/q(\psi)$. Substitution gives

$$\boxed{I = r(\psi(u))^2 (du^2 + dv^2).}$$

The metric is a positive scalar multiple of the Euclidean metric. This scales all lengths at a point by the same factor, so angles are unchanged. Thus u, v are isothermal parameters.

Problem 5, continued: points on the axis.

The angular coordinate is not defined on the axis of rotation. At a smooth pole, one more change of variables gives conformal coordinates that include the pole itself.

Use meridian arclength $s \geq 0$, measured from the pole, so

$$I = ds^2 + r(s)^2 dv^2.$$

Smoothness at the pole gives $r(s) = s a(s^2)$, where a is smooth and $a(0) = 1$. One way to see this is to write the surface near the pole as the rotational graph $z = H(x^2 + y^2)$. Then arclength is a smooth odd function of the radius, with derivative 1 at the origin, and so is its inverse.

Choose $\rho > 0$ by $d\rho/ds = \rho/r(s)$. Since $1/r(s) - 1/s$ is smooth and odd, integration yields $\rho = s b(s^2)$ with b smooth and positive. Therefore

$$I = \frac{r(s)^2}{\rho^2} (d\rho^2 + \rho^2 dv^2).$$

With $x = \rho \cos v$, $y = \rho \sin v$, this becomes

$$I = \frac{r(s)^2}{\rho^2} (dx^2 + dy^2).$$

The factor extends smoothly and positively across $\rho = 0$, giving a conformal coordinate disk at the pole.

Problem 7. Suppose we are given a surface element with $K < 0$. Show that this surface is a minimal surface if and only if the asymptotic curves at each point are perpendicular to one another.

Solution. An asymptotic direction is a nonzero tangent vector v with $\Pi(v, v) = 0$. At a fixed point choose an orthonormal principal basis e_1, e_2 . Since $K = \kappa_1\kappa_2 < 0$, we can label the curvatures so that $\kappa_1 > 0 > \kappa_2$.

For $v = xe_1 + ye_2$, the asymptotic condition is

$$\kappa_1 x^2 + \kappa_2 y^2 = 0.$$

Thus the two asymptotic lines are spanned by

$$v_+ = e_1 + ae_2, \quad v_- = e_1 - ae_2, \quad a = \sqrt{-\frac{\kappa_1}{\kappa_2}}.$$

They are perpendicular exactly when

$$0 = \langle v_+, v_- \rangle = 1 - a^2 = \frac{\kappa_1 + \kappa_2}{\kappa_2}.$$

Since $H = (\kappa_1 + \kappa_2)/2$, this is equivalent to $H = 0$ at the point. Applying the argument at every point proves the assertion, because a surface is minimal precisely when its mean curvature vanishes identically.

Problem 8. *More generally, show the following analog of 3.19: If an asymptotic curve of a surface element with $K < 0$ is a Frenet curve with torsion τ , then for the mean curvature we have the formula $H = \pm\tau \cot \varphi$, where φ is the angle between the two asymptotic curves.*

Solution. Parametrize the curve by arclength and use the Darboux frame of Problem 3. Since the curve is asymptotic, $\kappa_v = 0$, so

$$E'_1 = \kappa_g E_2, \quad E'_3 = -\tau_g E_2.$$

Its ordinary curvature is $\kappa = |\kappa_g| > 0$. On a connected interval let $\varepsilon = \text{sgn}(\kappa_g)$, which is constant. Its Frenet frame is therefore

$$T = E_1, \quad N = \varepsilon E_2, \quad B = T \times N = \varepsilon E_3.$$

Differentiating the last equation gives

$$B' = \varepsilon E'_3 = -\varepsilon \tau_g E_2 = -\tau_g N.$$

Comparison with the Frenet equation $B' = -\tau N$ shows that

$$\boxed{\tau = \tau_g.}$$

Problem 8, continued.

In the orthonormal tangent basis E_1, E_2 , the shape operator has matrix

$$[S] = \begin{pmatrix} 0 & \tau_g \\ \tau_g & 2H \end{pmatrix}.$$

Indeed, its first diagonal entry is $\kappa_v = 0$, its off-diagonal entry is τ_g , and its trace is $2H$. In particular, $K = -\tau_g^2$, recovering $\tau^2 = -K$.

Choose a unit vector in the other asymptotic direction and write

$$V = \cos \varphi E_1 + \sin \varphi E_2.$$

The two asymptotic lines are distinct, so $\sin \varphi \neq 0$. The equation $\Pi(V, V) = 0$ becomes

$$0 = 2\tau_g \cos \varphi \sin \varphi + 2H \sin^2 \varphi.$$

Dividing by $2 \sin^2 \varphi$ gives

$$\boxed{H = -\tau_g \cot \varphi = -\tau \cot \varphi.}$$

Here φ is measured in the oriented basis E_1, E_2 . For an unoriented angle between the two lines, the choice of which supplementary angle to use changes the sign of its cotangent. The orientation-independent statement is therefore $H = \pm \tau \cot \varphi$, as claimed.

Problem 9. *The Mercator projection*

$$f(u, \varphi) = \frac{1}{\cosh u}(\cos \varphi, \sin \varphi, \sinh u)$$

is a parametrization of the surface of the sphere without the north and the south pole. Show that this parametrization is angle preserving, i.e., that u, φ are isothermal parameters. In the science of cartography, a map with this property is referred to as angle preserving or conformal. Think of it as a circular cylinder touching the sphere along the equator.

Solution. Put $q = 1/\cosh u$ and $t = \tanh u$. Then $q^2 + t^2 = 1$, and

$$f = (q \cos \varphi, q \sin \varphi, t).$$

In particular, $\|f\|^2 = q^2 + t^2 = 1$. Differentiation gives

$$\begin{aligned} f_u &= (-qt \cos \varphi, -qt \sin \varphi, q^2), \\ f_\varphi &= (-q \sin \varphi, q \cos \varphi, 0). \end{aligned}$$

Problem 9, continued.

Taking inner products, we find

$$\begin{aligned}g_{11} &= q^2 t^2 + q^4 = q^2(t^2 + q^2) = q^2, \\g_{12} &= q^2 t \cos \varphi \sin \varphi - q^2 t \sin \varphi \cos \varphi = 0, \\g_{22} &= q^2(\sin^2 \varphi + \cos^2 \varphi) = q^2.\end{aligned}$$

Consequently,

$$\boxed{I = \frac{du^2 + d\varphi^2}{\cosh^2 u}}.$$

This is a positive scalar multiple of the Euclidean metric, so f is regular and conformal at every finite value of u . For example, for nonzero coordinate vectors a, b ,

$$\frac{\langle Df(a), Df(b) \rangle}{\|Df(a)\| \|Df(b)\|} = \frac{q^2 \langle a, b \rangle}{q\|a\| q\|b\|} = \frac{\langle a, b \rangle}{\|a\| \|b\|}.$$

Thus the angle between any two directions is preserved.

As $u \rightarrow \pm\infty$, the height $\tanh u$ approaches ± 1 , explaining the missing poles. The longitude φ is periodic; restricting it to a suitable interval gives a local coordinate chart.

Problem 18. *Prove that the ellipsoid of rotation defined by the equation*

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} = 1$$

is a Weingarten surface satisfying the equation $\kappa_1 = (a^4/c^2)(\kappa_2)^3$. Conversely, every compact surface of rotation with a constant ratio between κ_1 and κ_2^3 is an ellipsoid. Hint for the converse direction: The same approach as in 3.27.

Solution. We label the principal curvatures by direction: κ_1 is the meridian curvature and κ_2 the parallel curvature. A Weingarten surface satisfies a fixed relation between its principal curvatures. Assume $a, c > 0$, and parametrize by

$$f(\theta, v) = (a \sin \theta \cos v, a \sin \theta \sin v, c \cos \theta),$$

where $0 < \theta < \pi$. Set

$$L = \sqrt{a^2 \cos^2 \theta + c^2 \sin^2 \theta}.$$

Then $E = L^2$, $F = 0$, $G = a^2 \sin^2 \theta$. Choose the inward unit normal

$$\nu = -\frac{1}{L}(c \sin \theta \cos v, c \sin \theta \sin v, a \cos \theta).$$

Problem 18, continued.

The second fundamental form coefficients are

$$\begin{aligned}b_{11} &= \langle f_{\theta\theta}, \nu \rangle = \frac{ac}{L}, \\b_{12} &= \langle f_{\theta v}, \nu \rangle = 0, \\b_{22} &= \langle f_{vv}, \nu \rangle = \frac{ac \sin^2 \theta}{L}.\end{aligned}$$

Since both fundamental forms are diagonal, the coordinate directions are principal, with

$$\kappa_1 = \frac{b_{11}}{E} = \frac{ac}{L^3}, \quad \kappa_2 = \frac{b_{22}}{G} = \frac{c}{aL}.$$

Therefore

$$\boxed{\kappa_1 = \frac{a^4}{c^2} \kappa_2^3.}$$

The identity extends to the poles by continuity. Reversing the normal changes the signs of both curvatures and preserves this cubic relation.

For the converse, a compact surface is understood to be connected, regular, and without boundary. The assertion that the ratio is constant means $\kappa_2 \neq 0$ and $\kappa_1 = C\kappa_2^3$ for a constant C .

Problem 18, continued: the converse.

Away from the axis, write a unit-speed meridian as $(r(s), z(s))$. Up to the normal orientation, its parallel curvature is $z'(s)/r(s)$. Thus $\kappa_2 \neq 0$ implies $z'(s) \neq 0$: height can be used as a local parameter.

Write

$$X(z, v) = (r(z) \cos v, r(z) \sin v, z), \quad r(z) > 0.$$

Using subscripts for differentiation with respect to z , choose

$$v = \frac{(-\cos v, -\sin v, r_z)}{\sqrt{1 + r_z^2}}.$$

The fundamental forms are diagonal, with

$$\begin{aligned} E &= 1 + r_z^2, & G &= r^2, \\ b_{11} &= -\frac{r_{zz}}{\sqrt{1 + r_z^2}}, & b_{22} &= \frac{r}{\sqrt{1 + r_z^2}}. \end{aligned}$$

Consequently,

$$\kappa_1 = -\frac{r_{zz}}{(1 + r_z^2)^{3/2}}, \quad \kappa_2 = \frac{1}{r\sqrt{1 + r_z^2}}.$$

The assumed curvature relation reduces to the simple equation

$$\boxed{r_{zz} = -\frac{C}{r^3}}.$$

Problem 18, continued: solving the equation.

Multiplying the equation by $2r_z$ shows that

$$\frac{d}{dz} \left(r_z^2 - \frac{C}{r^2} \right) = 0.$$

Hence $r_z^2 - C/r^2 = A$ for a constant A . It follows that

$$(r^2)_{zz} = 2r_z^2 + 2rr_{zz} = 2A,$$

so

$$r(z)^2 = Az^2 + Bz + D.$$

Along a meridian, height is strictly monotone until the axis is reached. Compactness and absence of boundary force both ends to meet the axis: an endpoint with $r > 0$ could be continued, and an off-axis height extremum would have $z' = 0$. Thus this quadratic is positive between two finite zeros.

In particular, $A < 0$. Completing the square, write

$$r(z)^2 = a^2 - \frac{a^2}{c^2}(z - z_0)^2, \quad a, c > 0.$$

Rotating this meridian and including its two smooth poles gives

$$\boxed{\frac{x^2 + y^2}{a^2} + \frac{(z - z_0)^2}{c^2} = 1.}$$

Thus the surface is an ellipsoid of rotation, up to a rigid motion. Substitution also gives $C = a^4/c^2 > 0$.

Problem 19. Calculate the functions $\varphi_1, \varphi_2, \varphi_3$ for the Henneberg surface and verify the relation $\varphi_1^2 + \varphi_2^2 + \varphi_3^2 = 0$.

Solution. The parametrization in the text is

$$\begin{aligned} f_1(u, v) &= 2 \sinh u \cos v - \frac{2}{3} \sinh(3u) \cos(3v), \\ f_2(u, v) &= 2 \sinh u \sin v + \frac{2}{3} \sinh(3u) \sin(3v), \\ f_3(u, v) &= 2 \cosh(2u) \cos(2v). \end{aligned}$$

Following §3.31, put $z = u + iv$ and define

$$\varphi_j(z) = \frac{\partial f_j}{\partial u} - i \frac{\partial f_j}{\partial v}.$$

For example,

$$\begin{aligned} (\partial_u - i\partial_v)(\sinh u \cos v) &= \cosh u \cos v + i \sinh u \sin v \\ &= \cosh z, \\ (\partial_u - i\partial_v)(\sinh u \sin v) &= \cosh u \sin v - i \sinh u \cos v \\ &= -i \sinh z. \end{aligned}$$

The triple-angle terms are differentiated in exactly the same way, with a factor of 3 from the chain rule.

Problem 19, continued.

Differentiating all three components gives

$$\begin{aligned}\varphi_1(z) &= 2 \cosh z - 2 \cosh(3z), \\ \varphi_2(z) &= -2i \sinh z - 2i \sinh(3z), \\ \varphi_3(z) &= 4 \sinh(2z).\end{aligned}$$

Use the addition identities

$$\begin{aligned}\cosh z - \cosh(3z) &= -2 \sinh(2z) \sinh z, \\ \sinh z + \sinh(3z) &= 2 \sinh(2z) \cosh z.\end{aligned}$$

They give the useful factorization

$$(\varphi_1, \varphi_2, \varphi_3) = 4 \sinh(2z)(-\sinh z, -i \cosh z, 1).$$

Consequently,

$$\begin{aligned}\varphi_1^2 + \varphi_2^2 + \varphi_3^2 &= 16 \sinh^2(2z)(\sinh^2 z - \cosh^2 z + 1) \\ &= 0.\end{aligned}$$

These are complex squares, not absolute squares. The functions are holomorphic, and the identity proves conformality wherever the parametrization is regular. Their common zeros are $z = k\pi i/2$, $k \in \mathbb{Z}$; at these points the differential vanishes, so they must be excluded from regular surface patches.