MA 563: Differential Geometry

Group Project Handout

Overview

The project is an opportunity to learn and explain a piece of differential geometry that extends the topics presented in lecture. The project is worth 25% of your final course grade as indicated in the syllabus.

Groups and important dates

Students will work in **groups of two or three**. Each group will submit one joint paper and give one joint presentation. All group members are expected to understand the entire project, not only the portion they drafted or presented.

Partner and topic selection	Oct 8
Preliminary bibliography/proposal	Oct 22
Written report due	Dec 1
Presentations	Dec 3,8

Topic assignments and the process for proposing an alternative topic are TBA.

Written report

The report should be a **6-10 page mathematical expository paper**. This will not be a collection of notes or a sequence of copied definitions. It should tell a coherent mathematical story for your peers taking MA 563.

Every report must include:

- a clear introduction that states the central question and explains how the topic extends the course;
- precise definitions and notation;
- at least one substantial theorem, with a complete proof or a substantial proof sketch whose omitted steps are identified;
- at least one worked example, including an explicit computation;
- a bibliography and citations for all ideas, proofs, figures, and computations taken from sources.

The report should be prepared using LaTeX and submitted as a printed PDF. If you have any questions about using LaTeX on your device please reach out to me early in the semester and I will be happy to guide you through the basic usage.

Presentation

Each group will give a **20 minute presentation**, followed by brief questions. The presentation should select the most illuminating part of the report rather than trying to compress every detail into a few minutes.

Both partners must speak. The presentation should:

- motivate the topic and connect it to material from MA 563;
- state the principal definitions and result clearly;
- explain one proof idea, example, or calculation in meaningful detail;
- use readable board work or visual aids (a slide presentation is acceptable);
- finish within the allotted time.

Topic choices

The scope described under each topic is a starting point. A successful project should develop a focused question rather than attempt to cover every item listed.

1. **Vector bundles on surfaces and the tangent bundle.** Introduce a rank-r real vector bundle

   $$\pi\colon E \longrightarrow S$$

   using local trivializations and transition functions. Discuss sections and trivial bundles, with $TS \to S$ as the principal example. Explain why TS^2 is not trivial and connect this obstruction to the hairy-ball theorem or Euler characteristic.

2. **The Poincaré–Hopf theorem and the hairy-ball theorem.** Define the index $\mathrm{ind}_p(X)$ of an isolated zero of a vector field and explain

   $$\boxed{\sum_{X(p)=0} \mathrm{ind}_p(X) = \chi(S).}$$

   Deduce the hairy-ball theorem for S^2, and compare Poincaré–Hopf with Gauss–Bonnet as two ways of detecting the Euler characteristic.

3. **Differential forms, Stokes' theorem, and curvature.** Introduce differential forms on a surface, the exterior derivative

   $$\mathrm{d}\colon \Omega^k(S) \longrightarrow \Omega^{k+1}(S),$$

   and Stokes' theorem. Reformulate part of the course in this language.

4. **Minimal surfaces: the catenoid and helicoid.** Develop the geometry of surfaces satisfying $H = 0$. Explain the relation to the first variation of area, compute the mean curvature of the catenoid and helicoid, and prove that both are minimal. An extension may discuss their associated family of locally isometric minimal surfaces.

5. **Geodesics on surfaces of revolution and Clairaut's relation.** Derive the geodesic equations for a surface of revolution and prove Clairaut's relation

   $$\boxed{r \sin\theta = \text{constant.}}$$

Use it to study geodesics on a sphere, cylinder, cone, or torus, making the meaning of the angle θ precise.

6. **Hyperbolic geometry and the hyperbolic plane.** Study a model such as the upper half-plane

$$\mathbb{H}^2 = \{(x, y) \in \mathbf{R}^2 : y > 0\}, \qquad ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

   Compute $K = -1$, describe its geodesics, and derive or explain the hyperbolic triangle formula

$$\mathrm{Area}(\triangle) = \pi - (\alpha + \beta + \gamma).$$

7. **The exponential map, conjugate points, and geodesics on the sphere.** Develop

$$\exp_p : T_pS \longrightarrow S$$

   beyond its treatment in lecture and compute it explicitly for S^2. Explain conjugate points, why the antipodal point is special, and the difference between locally minimizing and globally minimizing geodesics. A further extension may discuss the cut locus.

8. **Complete surfaces and the Hopf–Rinow theorem.** Distinguish metric completeness from geodesic completeness and state the Hopf–Rinow theorem for surfaces. Contrast $\mathbf{R}^2$ with an open Euclidean disk, and study examples such as the sphere and hyperbolic plane. The full theorem need not be proved, but the report must prove a meaningful special case or substantial ingredient.

9. **Moving frames and the Gauss–Codazzi equations.** Use an orthonormal moving frame e_1, e_2, N and connection forms to derive structure equations. Show how the compatibility condition $d^2 = 0$ leads to the Gauss and Codazzi equations. Explain how this formalism organizes coordinate computations from the course.

10. **Curvature and topology: consequences of Gauss–Bonnet.** Explore consequences of

$$\boxed{\int_S K\, dA = 2\pi\chi(S).}$$

   For a closed orientable surface of genus g, use $\chi(S) = 2 - 2g$ to obtain

$$\int_S K\, dA = 4\pi(1 - g).$$

   Investigate restrictions on the sign of curvature for the sphere, torus, and higher-genus surfaces. An extension may connect this discussion to classification or uniformization.

Sources and academic integrity

Use at least two serious mathematical sources (outside of, or in addition to, our textbook), at least one of which should be a textbook, monograph, or refereed set of lecture notes. General-reference websites may help with orientation but do not by themselves constitute an adequate bibliography. Cite sources at the point where they are used. Copying text, proofs, calculations, or figures without attribution is not acceptable.

Grading rubric

The rubric below totals 25 points, corresponding to the project's 25% share of the course grade.

Category	Standard	Points
Mathematical correctness and depth	Definitions, statements, computations, and arguments are correct; the project develops a focused topic beyond the core lectures.	6
Exposition and organization	The report has a coherent narrative, clear notation, appropriate motivation, and readable mathematical prose.	4
Theorem and proof	At least one substantial theorem is treated with a complete proof or a clearly identified and mathematically meaningful proof sketch.	3
Example, figures, and sources	A worked example or construction illuminates the theory; figures are effective when used; sources are appropriate and cited consistently.	2
	Written report subtotal	*15*
Presentation clarity and structure	The talk is motivated, logically organized, legible, and understandable to classmates.	3
Mathematical substance	The talk communicates a central definition/result and explains a proof idea, example, or computation accurately.	2
Delivery and timing	All partners contribute meaningfully; visual aids or board work help; the group stays within 12–15 minutes.	2
	Presentation subtotal	*7*
Individual understanding	Each student demonstrates command of the full project when answering questions or discussing the work.	3
	Total	**25**

Normally, the report and presentation receive shared group scores, while the individual-understanding score is assigned separately. If the evidence shows a substantial imbalance in participation, the instructor may adjust individual scores after discussing the issue with the group.