Differential forms, Stokes' theorem, and topology

Local potentials and global obstructions

Written solutions. Work in pairs and choose one topic sheet. Submit one joint **5–10 page, cleanly typeset PDF generated using LaTeX**, organized by problem number. Address all five problems. Write complete mathematical sentences, define notation, justify your arguments, and include useful examples or figures. This is a polished write-up of solutions, not a separate expository essay. Cite sources where used and include a short bibliography. Give proofs unless a result is explicitly permitted without proof; distinguish conjecture and numerical evidence from proof. Both partners should understand the entire submission.

Presentation. Give a joint **20-minute presentation**, followed by brief questions. Both partners should speak. Explain a central result, one proof idea, and an illuminating example rather than trying to present every solution. Project deadlines and grading weight remain as previously announced.

Guiding question: How can derivatives and integrals detect a hole in a surface?

Problem 1. Differential forms and changes of coordinates

A smooth *one-form* on a surface assigns a linear functional $\omega_p : T_pS \to \mathbb{R}$ smoothly to each point. In coordinates it has the form $P\,du + Q\,dv$. A *two-form* assigns an alternating bilinear functional; locally it is $A\,du \wedge dv$, where

$$(du \wedge dv)(V, W) = du(V)dv(W) - du(W)dv(V).$$

For a smooth map F, define the pullback by $(F^*\omega)_p(V) = \omega_{F(p)}(dF_pV)$, and similarly for two-forms, using dF_p in both slots.

(a) Derive the coordinate-change rules for one-forms and two-forms. Explain why a two-form acquires a Jacobian determinant, not its absolute value, under a change of coordinates.

(b) For polar coordinates $F(r, \theta) = (r\cos\theta, r\sin\theta)$, compute $F^*(dx \wedge dy)$ and the pullback of

$$\eta = \frac{-y\,dx + x\,dy}{x^2 + y^2}.$$

Explain why the notation $\eta = d\theta$ is meaningful locally but does not yet give a globally defined real-valued angle function.

Problem 2. Closed does not always mean exact

For a smooth function f, let $df = f_x\,dx + f_y\,dy$, and define

$$d(P\,dx + Q\,dy) = (Q_x - P_y)\,dx \wedge dy.$$

A one-form is *closed* if $d\omega = 0$ and *exact* if $\omega = df$ for a globally defined smooth function. These definitions are coordinate-independent; this fact may be quoted. For $c : [a, b] \to S$, define $\int_c \omega = \int_a^b \omega_{c(t)}(\dot{c}(t))\,dt$. Prove that exact forms are closed and that their integral around every closed curve is zero. Show that η from Problem 1 is closed on $\mathbb{R}^2 \setminus \{0\}$, but has integral 2π around the positively oriented unit circle. Conclude that it is not exact. Find a different closed one-form on this punctured plane that *is* exact, and identify its potential.

Problem 3. Why a local potential always exists

An open set $U \subset \mathbb{R}^2$ is *star-shaped about* 0 if the segment from 0 to each point of U lies in U. For a closed form $\omega = P\,dx + Q\,dy$ on such a set, define

$$f(x,y) = \int_0^1 \big(xP(tx,ty) + yQ(tx,ty)\big)\,dt.$$

Prove that $df = \omega$. You may differentiate under the integral sign; use $P_y = Q_x$ to recognize a derivative with respect to t. Deduce that every closed one-form on a surface is locally exact, by shrinking a coordinate neighborhood to a disk. Explain precisely why this local statement does not contradict Problem 2. Your proof should also cover a convex open set not containing the origin, after translating the formula to a chosen base point.

Problem 4. Stokes' theorem and a global obstruction

You may quote Stokes' theorem in the following form: for a compact oriented smooth surface S with boundary and a smooth one-form ω,

$$\int_S d\omega = \int_{\partial S} \omega,$$

with the induced boundary orientation. Integrals of two-forms are computed in oriented coordinates; the usual change-of-variables theorem makes them well-defined.

(a) Apply Stokes' theorem to a circular annulus and η. Explain the signs of the two boundary integrals, and why applying the theorem instead to the full disk is invalid.

(b) On an oriented surface $S \subset \mathbb{R}^3$ with unit normal ν, define the area form by $\Omega_p(V,W) = \langle \nu(p), V \times W\rangle$. Show that Ω cannot equal $d\omega$ for any globally defined one-form on a nonempty compact oriented surface without boundary. Contrast this with the area form of the Euclidean plane, which is exact.

Problem 5. Every obstruction on the punctured plane is a period

Prove that every smooth closed one-form ω on $\mathbb{R}^2 \setminus \{0\}$ can be written

$$\omega = df + c\eta, \qquad c = \frac{1}{2\pi}\int_{S^1} \omega,$$

where the circle is counterclockwise and f is globally defined and smooth. Thus ω is exact if and only if its circle integral vanishes.

Suggested route. Pull back by $q : (0,\infty) \times \mathbb{R} \to \mathbb{R}^2 \setminus \{0\}$, $q(r,t) = (r\cos t, r\sin t)$. You may quote $d(q^*\omega) = q^*(d\omega)$; Problem 3 gives $q^*\omega = dF$. Show that $F(r,t+2\pi) - F(r,t)$ is constant and equals the circle integral. Deduce that $F(r,t) - ct$ descends to the required function f. Interpret the result as saying that closed one-forms, considered equal when they differ by an exact form, form a one-dimensional vector space.

Presentation suggestion. Introduce η and explain how a form can have local potentials but no global potential. Use Stokes' theorem or Problem 5 to make the topological obstruction precise.

Starting point. Lorenzo Sadun, *Lecture Notes on Differential Forms*, especially the introductory discussions of exterior differentiation, Stokes' theorem, and closed versus exact forms. The problems above are self-contained once the explicitly quoted results are accepted.