Geodesics on surfaces of revolution

Clairaut's relation and the geometry of geodesics

Written solutions. Work in pairs and choose one topic sheet. Submit one joint **5–10 page, cleanly typeset PDF generated using LaTeX**, organized by problem number. Address all five problems. Write complete mathematical sentences, define notation, justify your arguments, and include useful examples or figures. This is a polished write-up of solutions, not a separate expository essay. Cite sources where used and include a short bibliography. Give proofs unless a result is explicitly permitted without proof; distinguish conjecture and numerical evidence from proof. Both partners should understand the entire submission.

Presentation. Give a joint **20-minute presentation**, followed by brief questions. Both partners should speak. Explain a central result, one proof idea, and an illuminating example rather than trying to present every solution. Project deadlines and grading weight remain as previously announced.

Guiding question: What can a conserved quantity tell us without explicitly solving the geodesic equations?

Dots denote differentiation in an arbitrary parameter t; primes on a geodesic denote arclength s. Write $r_u = dr/du$ for derivatives of functions of the meridian coordinate. Angular coordinates are understood locally, using real lifts when differentiating.

Problem 1. The metric and geodesic equations

Consider a surface of revolution

$$f(u,\theta) = (r(u)\cos\theta, r(u)\sin\theta, z(u)), \qquad r > 0, \qquad r_u^2 + z_u^2 = 1.$$

Thus u measures arclength along a meridian.

(a) Compute the first fundamental form. Show that $e_u = f_u$ and $e_\theta = f_\theta/r$ are an orthonormal tangent frame. Give an example of a surface with this parametrization.

(b) You may use

$$\Gamma_{ij}^k = \frac{1}{2}\sum_\ell g^{k\ell}(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij}), \qquad \ddot{x}^k + \sum_{ij}\Gamma_{ij}^k \dot{x}^i \dot{x}^j = 0.$$

Compute the nonzero Christoffel symbols and derive

$$\ddot{u} - r r_u \dot{\theta}^2 = 0, \qquad \ddot{\theta} + 2\frac{r_u}{r}\dot{u}\dot{\theta} = 0.$$

(c) Prove that meridians traversed at constant speed are geodesics. Determine exactly which nonconstant parallels $u = u_0$, traversed at constant angular speed, are geodesics.

Problem 2. Clairaut's relation and turning points

Let $c(s) = f(u(s), \theta(s))$ be a unit-speed geodesic.

(a) Prove that $C = r(u)^2\theta'$ is constant. If a locally continuous oriented angle α is defined by $c' = \cos\alpha\, e_u + \sin\alpha\, e_\theta$, show that

$$r(u)\sin\alpha = C, \qquad u'^2 = 1 - \frac{C^2}{r(u)^2}.$$

Interpret the first equality geometrically.

(b) Show that the geodesic can enter only the region $r \geq |C|$. If $C \neq 0$, $r(u_0) = |C|$, and $r_u(u_0) \neq 0$, use the second-order equation to show that reaching u_0 gives a strict turning point of u.

(c) Find a unit-speed parallel that satisfies both displayed conservation laws but is not a geodesic. Explain why these necessary conditions do not by themselves give a general converse.

Problem 3. Unrolling a cone

Take $r(u) = au$, $z(u) = \sqrt{1 - a^2}\, u$, with $0 < a < 1$ and $u > 0$.

(a) Prove that

$$(u, \theta) \longmapsto \big(u\cos(a\theta), u\sin(a\theta)\big)$$

preserves the metric locally. Interpret this map by cutting and unrolling the cone, and determine the angle of the resulting sector.

(b) Describe geodesics as straight lines in the unrolled picture. You may use that local isometries carry geodesics to geodesics. Illustrate a geodesic crossing the chosen cut and explain why the cut is not a singularity of the cone.

(c) Use Clairaut's relation to show that only meridians can approach the missing vertex. Show that a unit-speed meridian reaches it in finite arclength. Is this punctured cone geodesically complete? Here completeness means that every geodesic extends for all real values of its affine parameter.

Problem 4. The sphere and the meaning of a coordinate singularity

For a sphere of radius R, use

$$r(u) = R\sin(u/R), \qquad z(u) = R\cos(u/R), \qquad 0 < u < \pi R.$$

(a) Describe the geodesics with $C = 0$, $0 < |C| < R$, and $|C| = R$. For the middle case, find the extreme values of u and the extreme latitudes in terms of $|C|$.

(b) You may use the result that nonconstant geodesic paths on a sphere are great circles. Reconcile your description with this result using a drawing or a geometric argument.

(c) A meridian reaches a pole in finite arclength. Explain why the conclusion about completeness differs from that for the cone. Distinguish a missing point of the surface from a missing point of the coordinate chart.

Problem 5. A qualitative picture of geodesics on a torus

Let $R > a > 0$, and set

$$r(u) = R + a\cos(u/a), \qquad z(u) = a\sin(u/a), \qquad u \in \mathbb{R}/(2\pi a\mathbb{Z}).$$

(a) Identify the inner and outer equators and prove that they are geodesics. Sketch $r(u)$ over one meridian period.

(b) Classify the allowed meridional motion in the four regimes

$$|C| < R - a, \quad R - a < |C| < R + a, \quad |C| = R + a, \quad |C| > R + a.$$

Which geodesics wind around the meridian, which oscillate near the outer equator, and which are impossible? Use the geodesic equations and turning-point criterion, not just the inequality $r \geq |C|$.

(c) Describe what is special about the remaining value $|C| = R - a$. You may quote that a nonconstant geodesic with this value can approach the inner equator asymptotically but cannot reach it in finite arclength. Explain why this is consistent with uniqueness for the geodesic equations.

(d) Draw representative paths or phase portraits for three different regimes. Explain why returning to the same meridian coordinate does not by itself mean that the geodesic is closed. Numerical illustrations are welcome but are not required.

Suggested presentation. Derive Clairaut's relation, use an unrolled cone to give geometric intuition, and explain the torus's different types of meridional motion.

Starting points. Kühnel, Chapter 4, on geodesics and intrinsic geometry; Theodore Shifrin, *Differential Geometry: A First Course in Curves and Surfaces*, Chapter 2, Section 4, especially Clairaut's relation.