Lie groups: the geometry of rotations

Tangent spaces, Lie brackets, and the exponential

Written solutions. Work in pairs and choose one topic sheet. Submit one joint **5–10 page, cleanly typeset PDF generated using LaTeX**, organized by problem number. Address all five problems. Write complete mathematical sentences, define notation, justify your arguments, and include useful examples or figures. This is a polished write-up of solutions, not a separate expository essay. Cite sources where used and include a short bibliography. Give proofs unless a result is explicitly permitted without proof; distinguish conjecture and numerical evidence from proof. Both partners should understand the entire submission.

Presentation. Give a joint **20-minute presentation**, followed by brief questions. Both partners should speak. Explain a central result, one proof idea, and an illuminating example rather than trying to present every solution. Project deadlines and grading weight remain as previously announced.

Guiding question. How does the geometry of a space of symmetries turn the familiar cross product into a Lie bracket?

Starting definitions. A *Lie group* is a smooth manifold equipped with a group operation for which multiplication and inversion are smooth. Here the group operation will be matrix multiplication. Write $M_3(\mathbb{R})$ for the 3×3 real matrices, I for the identity matrix, and set

$$O(3) = \{Q : Q^{\mathsf{T}}Q = I\}, \qquad SO(3) = \{Q \in O(3) : \det Q = 1\}.$$

Tangent vectors to a matrix manifold are derivatives $\dot{Q}(0)$ of curves in that manifold. You may use the regular-level-set theorem: if dF is surjective along $F^{-1}(0)$, then this level set is a smooth submanifold with tangent space $\ker dF$.

Problem 1. The manifold of rotations

Regard the symmetric 3×3 matrices as a six-dimensional vector space, and consider

$$F : M_3(\mathbb{R}) \longrightarrow \mathrm{Sym}_3(\mathbb{R}), \qquad F(Q) = Q^{\mathsf{T}}Q - I.$$

(a) Compute $dF_Q(V)$ and prove it is surjective when $Q \in O(3)$. Deduce that $O(3)$ is a three-dimensional manifold and that $SO(3)$ is an open subset of it. Verify that $SO(3)$ is a Lie group.

(b) Show that

$$T_Q SO(3) = \{QA : A^{\mathsf{T}} = -A\}, \qquad \mathfrak{so}(3) := T_I SO(3) = \{A : A^{\mathsf{T}} = -A\}.$$

Explain why the tangent vectors here are matrices rather than vectors in the physical space on which the rotations act.

Hint. To obtain a prescribed symmetric matrix S as $dF_Q(V)$, try $V = QS/2$. On $O(3)$ the determinant takes only the values 1 and -1.

Problem 2. The cross product as a Lie bracket

For $w = (w_1, w_2, w_3) \in \mathbb{R}^3$, define

$$\widehat{w} = \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix}, \qquad \widehat{w}\, v = w \times v.$$

Prove that this is a vector-space isomorphism $\mathbb{R}^3 \to \mathfrak{so}(3)$ and that

$$[\widehat{w}, \widehat{z}] := \widehat{w}\widehat{z} - \widehat{z}\widehat{w} = \widehat{w \times z}.$$

Show that the matrix commutator is closed on $\mathfrak{so}(3)$ and satisfies the Jacobi identity $[A,[B,C]]+[B,[C,A]]+[C,[A,B]] = 0$. Together with bilinearity and antisymmetry, these properties define a *Lie algebra*. Describe the resulting bracket on $\mathbb{R}^3$ in words. *Hint.* Apply the commutator to a vector and use the vector triple product identity; for Jacobi, use associativity of matrix multiplication.

Problem 3. Recovering the Jacobi bracket of vector fields

For $P \in SO(3)$, let $L_P(Q) = PQ$ be *left translation*. A vector field X is *left invariant* if $X(PQ) = d(L_P)_Q X(Q)$. For $A \in \mathfrak{so}(3)$, set $X_A(Q) = QA$.

(a) Prove that X_A is a smooth, left-invariant tangent vector field and that every left-invariant vector field has this form, with $A = X(I)$.

(b) Use the convention $[X, Y] = DY \cdot X - DX \cdot Y$ to prove

$$[X_A, X_B](Q) = Q(AB - BA) = X_{[A,B]}(Q).$$

Thus explain how the algebraic bracket of Problem 2 agrees with the bracket of vector fields studied in class.

Hint. The formulas $Q \mapsto QA$ and $Q \mapsto QB$ extend to linear vector fields on the ambient vector space $M_3(\mathbb{R})$.

Problem 4. From infinitesimal rotations to actual rotations

Define $\exp(tA) = \sum_{k=0}^{\infty} t^k A^k / k!$. You may use termwise differentiation of this series and $\exp(sA)\exp(tA) = \exp((s+t)A)$.

(a) For $A \in \mathfrak{so}(3)$, prove that $\exp(tA) \in SO(3)$. Show that $Q(t) = Q_0 \exp(tA)$ solves $\dot{Q} = X_A(Q)$ with $Q(0) = Q_0$.

(b) For a unit vector u, derive Rodrigues' formula

$$\exp(t\widehat{u}) = I + (\sin t)\widehat{u} + (1 - \cos t)(\widehat{u})^2.$$

Explain geometrically why this is rotation by angle t about the oriented axis u. Include a labeled sketch or one concrete example.

Hint. Use $(\widehat{u})^2 v = \langle u, v\rangle u - v$. For the determinant in part (a), use continuity and its value at $t = 0$.

Problem 5. Why the order of rotations matters

For $A, B \in \mathfrak{so}(3)$, define the group commutator

$$C(t) = \exp(tA)\exp(tB)\exp(-tA)\exp(-tB).$$

Prove, by multiplying Taylor expansions through degree two, that

$$C(t) = I + t^2[A, B] + O(t^3).$$

Here the remainder notation means that each matrix entry is bounded in absolute value by a constant times $|t|^3$ for small t. For $A = \widehat{e}_1$ and $B = \widehat{e}_2$, identify the rotation axis and angle to leading order. Then find two explicit finite rotations about these axes that do not commute, and demonstrate this by following a single vector. Explain precisely in what sense the Lie bracket measures the failure of rotations to commute.

Suggested presentation. Build around one example: rotating about two different axes in opposite orders. Connect the visible discrepancy to tangent matrices, the cross product, and the second-order commutator formula.

Starting points for reading. John M. Lee, *Introduction to Smooth Manifolds*, second edition: Chapter 7 (Lie groups), Chapter 8 (Lie brackets and the Lie algebra of a Lie group), and the matrix exponential discussion in Chapter 20. The author's book page provides the contents and access information. No general classification of Lie groups or prior abstract algebra beyond the group axioms is needed.