Minimal surfaces: geometry and area

First variation, minimal graphs, and the catenoid–helicoid family

Written solutions. Work in pairs and choose one topic sheet. Submit one joint **5–10 page, cleanly typeset PDF generated using LaTeX**, organized by problem number. Address all five problems. Write complete mathematical sentences, define notation, justify your arguments, and include useful examples or figures. This is a polished write-up of solutions, not a separate expository essay. Cite sources where used and include a short bibliography. Give proofs unless a result is explicitly permitted without proof; distinguish conjecture and numerical evidence from proof. Both partners should understand the entire submission.

Presentation. Give a joint **20-minute presentation**, followed by brief questions. Both partners should speak. Explain a central result, one proof idea, and an illuminating example rather than trying to present every solution. Project deadlines and grading weight remain as previously announced.

Guiding question: What does vanishing mean curvature tell us about area, and what does it not tell us?

Throughout, $f : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ is a smooth immersion with chosen unit normal ν. Use the course conventions

$$g_{ij} = \langle f_i, f_j \rangle, \qquad h_{ij} = \langle f_{ij}, \nu \rangle, \qquad [L] = g^{-1}h, \qquad H = \tfrac{1}{2}\operatorname{tr}(L).$$

Thus $L = -D\nu \circ (Df)^{-1}$. A surface is *minimal* if $H = 0$; this condition does not depend on the choice of normal.

Problem 1. Mean curvature as the first variation of area

Let D be a compact smoothly bounded region contained in U, and let φ be a smooth function supported in the interior of D. Consider the normal variation $f_t = f + t\varphi\nu$, which is an immersion for sufficiently small $|t|$. Write $g(t)$ for its first fundamental form and $A(t) = \int_D \sqrt{\det g(t)}\, du\, dv$.

(a) Prove that

$$\left.\frac{d}{dt}\right|_{t=0} g_{ij}(t) = -2\varphi h_{ij}, \qquad A'(0) = -2\int_D H\varphi \, dA.$$

You may use the matrix identity $\frac{d}{dt}\det g = \det g\, \operatorname{tr}(g^{-1}\dot{g})$.

(b) Deduce that $H = 0$ precisely when area is stationary under every such normal variation, on every such D. For the converse, use a nonnegative smooth bump function near a point where H has a strict sign. Explain why this argument proves stationarity, not minimal area.

Problem 2. Minimal graphs and a nonlinear equation

For a graph $f(x, y) = (x, y, w(x, y))$, choose the upward normal. Derive the minimal surface equation

$$(1 + w_y^2)w_{xx} - 2w_x w_y w_{xy} + (1 + w_x^2)w_{yy} = 0.$$

A useful starting point is $g = I_2 + (\nabla w)(\nabla w)^T$ and $h = \operatorname{Hess}(w)/\sqrt{1 + |\nabla w|^2}$. Show that a harmonic function, satisfying $w_{xx} + w_{yy} = 0$, need not give a minimal graph: test $w(x, y) = xy$. Then verify that

$$w(x, y) = \log\left(\frac{\cos y}{\cos x}\right), \qquad |x|, |y| < \frac{\pi}{2},$$

does give a nonplanar minimal graph. Explain why the Laplace equation is nevertheless the first-order approximation to the minimal surface equation for a small graph $w = \varepsilon v$.

Problem 3. The catenoid and helicoid

Consider

$$X(u,v) = (\cosh v \cos u, \cosh v \sin u, v), \quad Y(u,v) = (\sinh v \sin u, -\sinh v \cos u, u).$$

Initially take (u,v) in a small open rectangle. Show that both are immersions with the same first fundamental form $g = \cosh^2 v\,(du^2 + dv^2)$. Use the common normal

$$\nu = (\operatorname{sech} v \cos u, \operatorname{sech} v \sin u, -\tanh v)$$

to compute their second fundamental forms and prove that both are minimal, with Gaussian curvature $K = -\operatorname{sech}^4 v$. Sketch the surfaces and explain the resulting local isometry. Why does the rule $X(u,v) \mapsto Y(u,v)$ not define a global map from the entire catenoid to the helicoid when u is considered modulo 2π?

Problem 4. An isometric family of minimal surfaces

An immersion is *conformally parametrized* if $g = \lambda^2(du^2 + dv^2)$ for some positive function λ. For such an immersion Z, prove

$$Z_{uu} + Z_{vv} = 2\lambda^2 H\nu.$$

First show that the left side is perpendicular to both Z_u and Z_v by differentiating the identities $|Z_u|^2 = |Z_v|^2$ and $\langle Z_u, Z_v\rangle = 0$. Conclude that conformal immersions are minimal exactly when all three coordinate functions are harmonic.

Now set $Z_t = \cos t\, X + \sin t\, Y$, where X, Y are from Problem 3. Using $Y_u = X_v$ and $Y_v = -X_u$, prove that every Z_t is a minimal immersion with the same first fundamental form. Describe the endpoints $t = 0$ and $t = \pi/2$. Does this calculation also prove that every intermediate surface is globally embedded? Explain what was and was not established.

Problem 5. Minimal need not mean area-minimizing

Take the catenoid piece $X(u,v)$ with u modulo 2π and $-T \le v \le T$, where $T > 0$. Its boundary consists of two circles of radius $\cosh T$ in the planes $z = \pm T$. Compute its area and compare it with the total area of the two flat disks spanning those circles:

$$A_{\text{cat}} = 2\pi(T + \sinh T \cosh T), \qquad A_{\text{disks}} = 2\pi \cosh^2 T.$$

Prove that the disks have smaller total area when $T = 1$. Explain why this does not contradict Problem 1, being explicit that this comparison allows a disconnected competitor and is not a small normal variation of the catenoid. Does this comparison alone settle the question of local stability under small boundary-fixing variations? Explain the distinction.

Presentation suggestion. Explain the first variation formula, then use either the catenoid–helicoid family or the area comparison to illustrate the geometry behind the definition of a minimal surface.

Starting point. Theodore Shifrin, *Differential Geometry: A First Course in Curves and Surfaces*, Chapter 2 for fundamental forms and Chapter 3, Section 4 for first variation. Check the choice of normal and shape-operator convention before importing formulas from any reference.