Smooth manifolds: from charts to embeddings

A problem-based project

Written solutions. Work in pairs and choose one topic sheet. Submit one joint **5–10 page, cleanly typeset PDF generated using LaTeX**, organized by problem number. Address all five problems. Write complete mathematical sentences, define notation, justify your arguments, and include useful examples or figures. This is a polished write-up of solutions, not a separate expository essay. Cite sources where used and include a short bibliography. Give proofs unless a result is explicitly permitted without proof; distinguish conjecture and numerical evidence from proof. Both partners should understand the entire submission.

Presentation. Give a joint **20-minute presentation**, followed by brief questions. Both partners should speak. Explain a central result, one proof idea, and an illuminating example rather than trying to present every solution. Project deadlines and grading weight remain as previously announced.

Guiding question: Can we define a smooth space intrinsically and then recover the embedded picture used in class?

All manifolds below are finite-dimensional and without boundary. A map between smooth manifolds is smooth when its coordinate representations are smooth; a diffeomorphism is a bijection smooth in both directions.

Problem 1. An intrinsic definition and a model example

A topological n-manifold is a Hausdorff, second-countable space locally homeomorphic to open subsets of $\mathbb{R}^n$. A chart is a homeomorphism from an open subset of the manifold onto an open subset of $\mathbb{R}^n$. A smooth atlas is a cover by charts $\varphi_i : U_i \to \varphi_i(U_i) \subset \mathbb{R}^n$ whose overlap maps are smooth with smooth inverse. A smooth structure is the maximal atlas containing these charts.

(a) Explain the topological conditions and specify the domain of $\varphi_j \circ \varphi_i^{-1}$. Explain what it means for an additional chart to be compatible.

(b) For $S^2 = \{(x, z) \in \mathbb{R}^2 \times \mathbb{R} : \|x\|^2 + z^2 = 1\}$, use

$$\varphi_N(x, z) = \frac{x}{1 - z}, \qquad \varphi_S(x, z) = \frac{x}{1 + z}.$$

Find the inverses and transition map, with their domains, and verify smooth compatibility. Explain why a single chart cannot cover S^2.

Problem 2. Recovering charts from embedded surfaces

Suppose $M \subset \mathbb{R}^N$ is locally a regular level set: near each point, $M = F^{-1}(0)$, where F is smooth on an open ambient neighborhood, takes values in $\mathbb{R}^{N-n}$, and has differential of rank $N - n$. Use the implicit function theorem to produce local graph charts. Show that their overlaps are smooth, so M inherits a smooth structure. You may use that subspaces of Euclidean space are Hausdorff and second-countable. Explain how the inverse charts correspond to the regular parametrizations used in class.

Problem 3. Tangent vectors without an ambient space

For smooth curves $\gamma : (-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = p$, declare two curves equivalent when their coordinate velocities at 0 agree in a chart at p.

(a) Use the chain rule to prove that this condition is independent of the chart. Show that the equivalence classes form an n-dimensional vector space T_pM, with addition and scalar multiplication defined through coordinate velocities.

(b) In the embedded situation of Problem 2, identify these classes with ambient curve velocities and prove $T_pM = \ker DF_p$. Include the concrete example $T_pS^2 = p^\perp$.

Problem 4. Immersions versus embeddings

For a smooth map $A : M \to \mathbb{R}^N$, define $dA_p[\gamma] = \frac{d}{dt} A(\gamma(t))\big|_{t=0}$. An immersion has injective differential; an embedding is an injective immersion that is a homeomorphism onto its image with the subspace topology.

(a) Find an immersion of a circle into $\mathbb{R}^2$ that is not an embedding. Prove the claims and explain what fails geometrically. You may use $t \mapsto (\sin t, \sin 2t)$, with t taken modulo 2π.

(b) Prove that an injective immersion from a compact manifold into $\mathbb{R}^N$ is an embedding. You may quote that a continuous bijection from a compact space to a Hausdorff space is a homeomorphism.

Problem 5. From an abstract compact manifold to an embedding

For compact M, you may quote the following bump-function result: there are finitely many charts (U_i, φ_i) and smooth $\rho_i \geq 0$, with compact supports contained in U_i, such that $\{\rho_i > 0\}$ cover M. Extend $\rho_i\varphi_i$ by zero outside U_i.

(a) Prove that

$$A(p) = \big(\rho_1(p), \rho_1(p)\varphi_1(p), \ldots, \rho_k(p), \rho_k(p)\varphi_k(p)\big) \in \mathbb{R}^{k(n+1)}$$

is smooth, injective, and an immersion, hence an embedding. *Hint:* At p, choose i with $\rho_i(p) > 0$. Use both the scalar and vector entries; if $dA_p(v) = 0$, first deduce $d\rho_i(v) = 0$.

(b) You may quote the Whitney embedding theorem in the form: every smooth n-manifold ($n \geq 1$) embeds in $\mathbb{R}^{2n+1}$. Explain how this and Problem 2 relate the intrinsic and embedded definitions. For the remaining direction, justify that the image of an embedding is locally a graph, hence a regular level set. *Hint:* Select a nonsingular $n \times n$ minor of the differential and apply the inverse function theorem; the embedding condition gives the required neighborhood in the image. Why does this not assert that every surface embeds in $\mathbb{R}^3$, or that the embedding preserves a prescribed metric?

Presentation focus. Explain the sphere's charts and the compact embedding construction, emphasizing the passage between the two definitions.

Starting points. John M. Lee, *Introduction to Smooth Manifolds*, second edition: charts, tangent spaces, embeddings, and bump functions. The author's page includes a free first chapter. Use Kühnel's discussion of submanifolds for the connection to class.