Vector bundles: local products, global twisting

A problem-based project

Written solutions. Work in pairs and choose one topic sheet. Submit one joint **5–10 page, cleanly typeset PDF generated using LaTeX**, organized by problem number. Address all five problems. Write complete mathematical sentences, define notation, justify your arguments, and include useful examples or figures. This is a polished write-up of solutions, not a separate expository essay. Cite sources where used and include a short bibliography. Give proofs unless a result is explicitly permitted without proof; distinguish conjecture and numerical evidence from proof. Both partners should understand the entire submission.

Presentation. Give a joint **20-minute presentation**, followed by brief questions. Both partners should speak. Explain a central result, one proof idea, and an illuminating example rather than trying to present every solution. Project deadlines and grading weight remain as previously announced.

Guiding question: How do tangent spaces fit together, and what can prevent them from forming a global product?

Work with smooth real vector bundles. Familiar charts on circles, spheres, and tori suffice for the examples; completing the manifolds sheet first is not required.

Problem 1. Vector bundles and tangent spaces

A *rank-r vector bundle* is a smooth map $\pi : E \to M$ whose fibers $E_p = \pi^{-1}(p)$ are r-dimensional vector spaces, with local trivializations

$$h_i : \pi^{-1}(U_i) \longrightarrow U_i \times \mathbb{R}^r.$$

These are diffeomorphisms preserving the base point and restricting to linear isomorphisms on fibers. The product bundle is $\varepsilon^r = M \times \mathbb{R}^r$.

(a) For an embedded surface M, set $TM = \{(p, v) : p \in M,\ v \in T_pM\}$. Use a parametrization $f : U \to M$ to construct a local trivialization by writing $v = a^1 f_1 + a^2 f_2$. Give the inverse map explicitly.

(b) On an overlap, write two parametrizations as $\widetilde{f} = f \circ \psi$. Show that the two column vectors of tangent coefficients satisfy $a = D\psi\, \widetilde{a}$. Explain why this is a linear change in each fiber even though ψ need not be linear.

(c) Explain the distinction between local triviality and the existence of a single global bundle isomorphism $TM \cong M \times \mathbb{R}^2$. A bundle isomorphism here means a diffeomorphism preserving the base point and linear on each fiber.

Problem 2. Sections and the criterion for triviality

A *section* is a smooth map $\sigma : M \to E$ satisfying $\pi \circ \sigma = \mathrm{id}_M$. A *global frame* consists of r sections forming a basis of every fiber.

(a) Prove that a rank-r bundle is trivial if and only if it has a global frame. Construct the isomorphism from a frame, and explain why its inverse is smooth using local trivializations.

(b) Deduce the special criterion for a line bundle: it is trivial if and only if it has a nowhere-zero section. Why does one nowhere-zero section not suffice to trivialize an arbitrary higher-rank bundle?

Problem 3. The Möbius line bundle

Define

$$E_{\mathrm{M}} = (\mathbb{R}\times\mathbb{R})/\big((t,a)\sim(t+2\pi,-a)\big), \qquad \pi([t,a]) = e^{it}\in S^1,$$

where the equivalence relation is generated by the displayed identification. Each fiber is an entire real line.

(a) Construct trivializations over two arcs covering the circle. Choose angular lifts so that the transition is $+1$ on one component of the overlap and -1 on the other.

(b) Show that smooth sections correspond to smooth functions $a:\mathbb{R}\to\mathbb{R}$ satisfying $a(t+2\pi) = -a(t)$. Use the intermediate value theorem and Problem 2 to prove that this bundle is nontrivial.

(c) Find a section that is not identically zero, describe its zeros, and sketch how it behaves around the circle. Distinguish "not identically zero" from "nowhere zero."

Problem 4. Tangent bundles that are trivial

Let $J(x,y) = (-y,x)$.

(a) Show that $p\mapsto Jp$ is a global tangent frame on S^1. Write the resulting isomorphism $S^1\times\mathbb{R}\to TS^1$.

(b) Regard the torus as $S^1\times S^1\subset\mathbb{R}^2\times\mathbb{R}^2$. Construct two everywhere independent tangent vector fields using J, and explicitly trivialize its tangent bundle.

(c) Find one further surface with a global tangent frame and justify your example. Explain why a drawing alone does not establish the existence of smooth, globally defined fields.

Problem 5. Why the sphere is different

You may use these facts about *degree*: every continuous map $S^2\to S^2$ has an integer degree; degree is unchanged under a homotopy; the identity has degree 1; and the antipodal map $p\mapsto -p$ has degree -1. A homotopy is a continuous family of maps parametrized by $[0,1]$.

(a) Suppose a nowhere-zero tangent vector field exists on S^2 and normalize it to a unit field X. Show that

$$H(p,t) = \cos(\pi t)\,p + \sin(\pi t)\,X(p)$$

defines a homotopy of maps $S^2\to S^2$. Identify its endpoint maps and derive a contradiction. This is the *hairy-ball theorem.*

(b) Deduce that TS^2 is not trivial. The sphere nevertheless has the global unit normal $p\mapsto p$. Explain why this does not contradict the theorem, and why orientability of a surface does not imply triviality of its tangent bundle.

Suggested presentation. Explain the frame criterion using the Möbius line bundle, then contrast the torus and the sphere. For the sphere, clearly identify the degree facts being quoted.

Starting points. Allen Hatcher, *Vector Bundles and K-Theory*, Section 1.1 (use smooth data in place of continuous data here). For the degree argument, see Hatcher, *Algebraic Topology*, Section 2.2, pp. 134–135.