HOMEWORK 10
NOT TO TURN IN

(1) Let $\{e_1, e_2, e_3\}$ be a basis for $\mathbf{R}^3$. Show that $e_1 \otimes e_2 \otimes e_3 \in \mathbf{R}^3 \otimes \mathbf{R}^3 \otimes \mathbf{R}^3$ is not equal to the sum of a completely anti-symmetric plus a completely symmetric three-tensor.

(2) Let $\omega \in \Omega^1(M)$ and $X, Y \in \mathrm{Vect}(M)$. Show that the two-form $\mathrm{d}\omega \in \Omega^2(M)$ satisfies

$$\mathrm{d}\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]). \tag{1}$$

(3) Given a k-form $\omega \in \Omega^k(M)$ and a vector field $X \in \mathrm{Vect}(M)$ define the k-form $L_X\omega \in \Omega^k(M)$ by the formula

$$(L_X\omega)|_p \stackrel{\text{def}}{=} \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}(\theta_t^*\omega)_p \tag{2}$$

where θ_t is the flow with respect to X (in particular, θ_t is a diffeomorphism from a neighborhood of p to a neighborhood of $\theta_t(p)$. This is called the *Lie derivative* of ω with respect to X.

Let $\omega \in \Omega^2(\mathbf{R}^4)$ be the two-form

$$\omega = x\mathrm{d}y + z\mathrm{d}w, \tag{3}$$

(where we use coordinates (x,y,z,w) for $\mathbf{R}^4$) and let X be the vector field

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + w\frac{\partial}{\partial w} \tag{4}$$

(a) Compute $L_X\omega$ in this example.
(b) Show by direct computation that

$$L_X\omega = \mathrm{d}\iota_X\omega + \iota_X\mathrm{d}\omega, \tag{5}$$