SOLUTIONS TO HOMEWORK 1

Problem 1. Let p be the point where the two lines intersect. For a point $q \neq p$ in the cross there clearly exists a neighborhood which is homeomorphic to an open interval. Therefore, if the cross were a topological manifold then its dimension would be one.

Any neighborhood of p will also be a cross, so it suffices to show that the there is no continuous map $\phi \colon + \to \mathbf{R}$ with the property that it induces a homeomorphism onto its image. It suffices to show that there is no continuous injection $\phi \colon + \to \mathbf{R}$.

Suppose that there is one. Let

$$U = + - \{p\} \tag{1}$$

be the open set obtained by removing the point p. Then the restriction of ϕ is a map $\phi|_U \colon U \to \mathbf{R} \setminus \phi(p)$. Note that U has four connected components, but $\mathbf{R} \setminus \phi(p)$ has two connected components. This contradicts the continuity of the injection $\phi|_U$.

Problem 2. This problem was meant to be subtle and can be clarified in the following way. The key point that the problem was attempting to probe was: *there is no smooth structure on $\vee$ such that the natural closed topological embedding $\vee \hookrightarrow \mathbf{R}^2$ is smooth.*[1] Viewing $\vee$ as the graph of the absolute value, this follows from the fact that the map $x \mapsto |x|$ is not smooth.

There does, however, exist a smooth structure on $\vee$ as many of you pointed out. To obtain one of them we can simply induce the standard smooth structure on $\mathbf{R}$ along the homeomorphism

$$\mathbf{R} \to \vee \tag{2}$$

defined by $x \mapsto (x, |x|)$.

Because the problem was stated unclearly, both solutions will be accepted.

[1] A topological embedding is a continuous map which is a homeomorphism onto its image.

Problem 3. It is clear that $\{U_\alpha \times V_\beta\}$ is a cover. We need to show that for all $\alpha, \beta, \alpha', \beta'$ that the charts

$$\phi_\alpha \times \psi_\beta \quad \text{and} \quad \phi_{\alpha'} \times \psi_{\beta'} \tag{3}$$

are smoothly compatible.

This relies on the following lemma.

Lemma 0.1. *Let U, V be open subsets of $\mathbf{R}^m, \mathbf{R}^n$ respectively and suppose $F\colon U \to \mathbf{R}^k$, $G\colon V \to \mathbf{R}^l$ are smooth functions. Then the map*

$$F \times G\colon U \times V \to \mathbf{R}^k \times \mathbf{R}^l \tag{4}$$

defined by $(x, y) \mapsto (F(x), G(y))$ is smooth.

Proof. Let $\mathbf{F} = F \times G$. First we show that $\mathbf{F}$ is C^1. Identifying $\mathbf{R}^k \times \mathbf{R}^l = \mathbf{R}^{k+l}$, it suffices to show that each of the component partial derivatives

$$\frac{\partial \mathbf{F}^i}{\partial x_j}, \quad i = 1, \ldots, k+l \quad j = 1, \ldots, m+n \tag{5}$$

exist and are continuous. For $i = 1, \ldots k$ this follows from the fact that F is C^1, for $i = k+1, \ldots l+k$ this follows from the fact that G is C^1. Iterating this we see that $\mathbf{F}$ is C^a for any integer $a > 0$. □

Now consider the composition

$$\phi_\alpha(U_\alpha \cap U_{\alpha'}) \times \psi_\beta(V_\beta \cap V_{\beta'}) \xrightarrow{\phi_\alpha^{-1} \times \psi_\beta^{-1}} (U_\alpha \cap U_{\alpha'}) \times (V_\beta \cap V_{\beta'}) \xrightarrow{\phi_{\alpha'} \times \psi_{\beta'}} \phi_{\alpha'}(U_\alpha \cap U_{\alpha'}) \times \psi_{\beta'}(V_\beta \cap V_{\beta'}), \tag{6}$$

which reads

$$(\phi_{\alpha'} \times \psi_{\beta'}) \circ (\phi_\alpha^{-1} \times \psi_\beta^{-1}) = (\phi_{\alpha'} \circ \phi_\alpha^{-1}) \times (\psi_{\beta'} \circ \psi_{\beta'}^{-1}). \tag{7}$$

Applying the lemma we see that this composition is smooth.