HOMEWORK 3
DUE OCTOBER 6

There are three problems to turn in.

(1) In this problem you will prove the 'local section theorem' stated as follows.

Theorem 0.1. *A smooth map $\pi\colon M \to N$ is a smooth submersion if and only if for every point $p \in M$ there exists an open set $U \subset N$ and a smooth map $\sigma\colon U \to M$ such that*

(a) $\pi \circ \sigma = \mathbb{1}_U$.

(b) $\sigma(q) = p$ *for some* $q \in U$.

(a) Let $m \geq n$ and $\pi_0\colon \mathbf{R}^m \to \mathbf{R}^n$ be the natural projection. Find a smooth map $\sigma_0\colon \mathbf{R}^n \to \mathbf{R}^m$ such that $\pi_0 \circ \sigma_0 = \mathbb{1}_{\mathbf{R}^n}$ and $\sigma_0(0) = 0$.

(b) Assume that π is a submersion. By the constant rank theorem there exists charts such that π has coordinate representation

$$\widehat{\pi}(x^1, \ldots, x^n, x^{n+1}, \ldots, x^m) = (x^1, \ldots, x^n).$$

Show that for $\epsilon > 0$ small enough that the image under $\widehat{\pi}$ of the cube

$$C_\epsilon = \{(x^1, \ldots, x^m) \mid |x^i| < \epsilon \text{ for all } i = 1, \ldots, m\}$$

is the cube

$$C'_\epsilon = \{(y^1, \ldots, y^m) \mid |y^j| < \epsilon \text{ for all } j = 1, \ldots, n\}.$$

(c) Use part (a) to construct the desired local section σ.

(d) Conversely, assume that there exists such a σ. Prove that π is a submersion.

(2) We have seen in class that the map $\pi\colon \mathrm{T}M \to M$ sending a tangent vector $v \in \mathrm{T}_pM$ at $p \in M$ to p is smooth. Show that π is a submersion.

(3) Suppose that $M \subset N$ is an embedded submanifold of dimension m.

(a) By assumption we can view $\mathrm{T}_pM \subset \mathrm{T}_pN$ as a linear subspace for any $p \in M$. Show that

$$\mathrm{T}_pM = \{v \in \mathrm{T}_pN \mid vf = 0 \text{ when } f \in C^\infty(N) \text{ satisfies } f|_M = 0\}.$$

(b) Let $N = \mathbf{R}^n$ and define

$$UM \overset{\text{def}}{=} \{(x, v) \in \mathrm{T}\mathbf{R}^n \mid x \in M, v \in \mathrm{T}_xM, \|v\| = 1\}.$$

Show that UM is an embedded submanifold of $\mathbf{R}^{2n}$ and compute its codimension.