HOMEWORK 8
SOLUTIONS

Problem 1. Let α, β be indices such that $U_\alpha \cap U_\beta \neq \emptyset$. The transition functions $g_{\alpha\beta}$ satisfy

$$\psi_\alpha \circ \psi_\beta^{-1}(p, v) = (p, g_{\alpha\beta}(p)v) \tag{1}$$

for all $p \in U_\alpha \cap U_\beta$ and $v \in E_p$. Suppose that $p \in U_\alpha \cap U_\beta \cap U_\gamma$. Then we have

$$\begin{aligned} \psi_\alpha \circ \psi_\gamma^{-1}(p, v) &= (\psi_\alpha \circ \psi_\beta^{-1}) \circ \psi_\beta \psi_\gamma^{-1}(p, v) \\ &= \psi_\alpha \circ \psi_\beta^{-1}(p, g_{\beta\gamma}(p)v) \\ &= (p, g_{\alpha_\beta}(p) g_{\beta\gamma}(p)v). \end{aligned}$$

But the left hand side is $(p, g_{\alpha\gamma}(p)v)$, which proves the assertion.

Problem 2.

(a) Let $\pi \colon V \to \mathbf{CP}^1$ be the map $(\ell, z_1, z_2) \mapsto \ell$. We define the trivialization maps using the standard cover $\{U_z, U_w\}$ from the previous homework. Define

$$\psi_z \colon \pi^{-1}(U_z) \to U_z \times \mathbf{C} \tag{2}$$

by $([1, w], z_1, z_2) \mapsto ([1, w], z_1)$ and

$$\psi_w \colon \pi^{-1}(U_w) \to U_w \times \mathbf{C} \tag{3}$$

by $([z, 1], z_1, z_2) \mapsto ([z, 1], z_2)$.

These trivialization satisfy $p_1 \circ \psi_{z,w} = \pi$, so we need to check that for each $q \in U_{z,w}$ that $\psi_{z,w}|_{V_q}$ is a linear isomorphism. Suppose $\ell = [1, w] \in U_z$, then $\psi_z|_{V_\ell}$ is the map $V_\ell = \ell \ni (z_1, z_2) \mapsto z_1$, which is certainly linear. Its inverse is $a \mapsto (a, aw)$, which is well-defined since $(a, aw) \in \ell = [1, w]$ for any $a \in \mathbf{C}$. Similarly $\psi_w|_{V_\ell}$ is an isomorphism for $\ell \in U_w$.

(b) Define

$$F \colon V \setminus \underline{0} \to \mathbf{C}^2 \setminus \{0\} \tag{4}$$

by the formula $(\ell, z_1, z_2) \mapsto (z_1, z_2)$ and

$$G \colon \mathbf{C}^2 \setminus \{0\} \to V \setminus \underline{0}. \tag{5}$$

by the formula $(z_1, z_2) \mapsto ([z_1, z_2], z_1, z_2)$. To see that G is well-defined just observe that by definition (z_1, z_2) is on the line $[z_1, z_2]$.

Being a projection F is certainly smooth. To see that G is smooth it suffices to use the standard charts on $\mathbf{CP}^1$. In the chart U_z, for example, G has the form

$$\widehat{G}(z_1, z_2) = (z_2/z_1, z_1, z_2) \tag{6}$$

which is certainly smooth.