November 15

Choose coordinate $\{x^i\}$ near $p \in M$. Then

$$T_pM \cong \operatorname{span}\left\{ \frac{\partial}{\partial x^i}\Big|_p \right\}.$$

Dual vector space:

$$T_p^*M \cong \operatorname{Span}\left\{ \lambda^i \right\}$$

when $\{\lambda^i\}$ is dual basis to $\left\{\frac{\partial}{\partial x^i}\Big|_p\right\}$.

Given $v \in T_pM$ have

$$v = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i}\Big|_p.$$

If we choose diffent coordinate $\{\tilde{x}^i\}$ then

$$\frac{\partial}{\partial x^i}\Big|_p = \frac{\partial \tilde{x}^j}{\partial x^i}(p) \frac{\partial}{\partial \tilde{x}^j}\Big|_p$$

Similarly, get new basis $\{\tilde{\lambda}^i\}$ for $T_p^* M$.

$$\left\langle \tilde{\lambda}^i, \frac{\partial}{\partial \tilde{x}^j} \right\rangle = \delta^i_j .$$

Sps that $\tilde{\lambda}^i = M^i_j \lambda^j$. Then

$$\delta^i_j = \left\langle M^i_l \lambda^l, \left(\left(\frac{\partial \tilde{x}}{\partial x} \right)^{-1} \right)^k_j \frac{\partial}{\partial x^k} \right\rangle .$$

$$= M^i_l \left(\left(\frac{\partial \tilde{x}}{\partial x} \right)^{-1} \right)^k_j \left\langle \lambda^l, \frac{\partial}{\partial x^k} \right\rangle$$

$$= M^i_k \left(\left(\frac{\partial \tilde{x}}{\partial x} \right)^{-1} \right)^k_j$$

$\Rightarrow$ $M = \dfrac{\partial \tilde{x}}{\partial x}$. In other words

$$\tilde{\lambda}^i = \sum_j \frac{\partial \tilde{x}^i}{\partial x^j} \lambda^j .$$

c.f.

$$\frac{\partial}{\partial \tilde{x}^i} = \sum_j \left(\left(\frac{\partial \tilde{x}}{\partial x} \right)^{-1} \right)^j_i \frac{\partial}{\partial x^j} .$$

Another way: If we write

$$T_pM \ni v = v^i \frac{\partial}{\partial x^i}\Big|_p$$

$$\Rightarrow \tilde{v}^i = \frac{\partial \tilde{x}^i}{\partial x^j} v^j$$

$$T_p^* M \ni \omega = \omega_i \lambda^i$$

$$\Rightarrow \omega_j = \frac{\partial \tilde{x}^i}{\partial x^j} \tilde{\omega}_i .$$

Cotangent bundle

Let

$$T^*M = \coprod_{p \in M} T_p^* M .$$

Thm: T^*M has the structure of a smooth vector bundle of rank $= \dim M$.

Pf : $\pi : T^{\alpha}M \to M$ is same as always.

To get triv's for $T^{\alpha}M$ we proceed as in the case of TM. Given coordinate chart (U, ϕ) for M, define

$$\psi : \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^n$$
$$\rotatebox{90}{$\in$}$$
$$\omega_i \lambda^i|_p \longmapsto (p; (\omega_1, \dots, \omega_n))$$

If $(\tilde{U}, \tilde{\phi})$ is anoth chart then

$$\psi \circ \tilde{\psi}^{-1}(p, (\omega_1, \dots, \omega_n))$$

$$= \left(p, \left(\frac{\partial \tilde{x}^i}{\partial x^1} \tilde{\omega}_i, \dots, \frac{\partial \tilde{x}^i}{\partial x^n} \tilde{\omega}_i\right)\right)$$

$$\frac{\partial \tilde{x}}{\partial x} : U \cap \tilde{U} \to GL(n, \mathbb{R})$$
$$p \longmapsto \left(\frac{\partial \tilde{x}^i}{\partial x^j}\right)$$ is smooth.

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More generally, given any v.b. $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$,

can define the dual bundle

$$\begin{matrix} E^* \\ \downarrow \\ M \end{matrix}$$

which has the property $E^*|_p = (E_p)^*$.

<u>Dfn</u>: A <u>1-form</u> is a smooth section

$$\omega \in \Gamma(M, T^*M) \overset{\text{dfn}}{=} \Omega^1(M)$$

- From functions to 1-forms. Given

$$f : M \to \mathbb{R}$$

we define $df \in \Omega^1(M)$ by

$$df_p(v) = vf.$$

Prop: df is a 1-form if $f \in C^\infty(M)$.

Pf: Need to see that $p \mapsto df_p$ is smooth. Suffices to show that for any smooth $X \in \mathrm{Vect}(M)$ that

$$df(X) \in C^\infty(M)$$

is smooth. But $df(X) = Xf$, and if X smooth then Xf smooth. $\square$

In coordinates

$$df_p = \frac{\partial f}{\partial x^i} \lambda^i \Big|_p .$$

In particular, if $f = x^j$ then

$$dx^j \Big|_p = \delta^j_i \, \lambda^i \Big|_p = \lambda^j \Big|_p .$$

For this reason we usually denote the coordinate basis for $T^0_p M$ by

$$\{ dx^i|_p \}$$

instead of $\{ \lambda^i|_p \}$. So,

$$\boxed{df_p = \frac{\partial f}{\partial x^i} dx^i|_p .}$$