Let

$$\Gamma(\mathcal{U}, E) = \left\{ \begin{array}{l} \text{set of local sections} \\ \text{defined on } \mathcal{U} \end{array} \right\}.$$

Then $\Gamma(\mathcal{U}, E)$ is naturally a vector space.

- $(s_1 + s_2)(p) = s_1(p) + s_2(p)$ in E_p.
- $(\lambda s)(p) = \lambda s(p)$

- A <u>local frame</u> for E on $\mathcal{U} \subset M$ is

a set of sections

$$\{s_1, \ldots, s_k\} \subset \Gamma(\mathcal{U}, E)$$

s.t. $\forall\, p \in \mathcal{U}$

$\{s_1(p), \ldots, s_k(p)\}$ is a basis for E_p.

A <u>global frame</u> is a local one when

$\mathcal{U} = M$. We've already discussed global frames for the tangent bundle.

<u>Ex</u>: Let $\{e_i\}$ be basis for $\mathbb{R}^k$. Then

there is a canonical frame for the trivial bundle

$$\begin{tikzcd} M \times \mathbb{R}^k \arrow[d] \\ M \arrow[u, bend right, "s_i"'] \end{tikzcd}$$

defined by
$$s_i : M \longrightarrow \mathbb{R}^k .$$
$$p \longmapsto e_i$$

$\leadsto$
In particular, any trivialization Ψ of a v.b. E over $\mathcal{U} \subset M$ defines a local frame on $\mathcal{U}$.

Prop: $\left\{ \begin{array}{l} \text{local frames for } E \\ \text{on } \quad U \subset M \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{local triv. for } E \\ \text{on } U \subset M \end{array} \right\}$

Pf: We need to show that a local frame determines a local triv.. If $\{s_i\}$ is local frame, then define

$$\chi : U \times \mathbb{R}^k \longrightarrow \pi^{-1}(U)$$
$$\cup$$
$$(p, v = (v^i)) \longmapsto v^i s_i(p).$$

Since $\{s_i(p)\}$ is basis it follows that χ is bijective.

Suffices to show ψ is local diffeomorphism. Given $q \in U$, choose nbd $V \subset U$ which has local triv:

$$\psi : \pi^{-1}(V) \longrightarrow V \times \mathbb{R}^k$$

Consider composition $\psi \circ x\big|_{V \times \mathbb{R}^k}$:

$$\begin{tikzcd} V \times \mathbb{R}^k \arrow[r, "x|_{V\times\mathbb{R}^k}"] \arrow[dr, "p_1"'] & \pi^{-1}(V) \arrow[r, "\psi"] \arrow[d, "\pi"] & V \times \mathbb{R}^k \arrow[dl, "p_1"] \\ & V \arrow[u, dashed, bend left, "s_i"] & \end{tikzcd}$$

For each i, have

$$\psi \circ s_i(p) = \left(p \,;\, s_i^1(p), \dots, s_i^k(p)\right)$$

for some $s_i^j : V \longrightarrow \mathbb{R}$ smooth.

$\Rightarrow$

$$\psi \circ x\big|_{V \times \mathbb{R}^k}\left(p, v = (v^i)\right)$$

$$= \left(p \,;\, v^i s_i^1(p), \dots, v^i s_i^k(p)\right)$$

which is clearly smooth.

Let $\quad \sigma_p = \left(s_i^j(p) \right) \in GL(k, \mathbb{R})$

then $\quad \sigma_p^{-1} = \left(t_i^j(p) \right) \in GL(k, \mathbb{R})$

for some coefficients $t_i^j(p)$. Then

$$(x \circ \psi)^{-1}(p, w = (w^i))$$

$$= \left(p ; w^i t_i^1(p), \dots, w^i t_i^k(p) \right)$$

which is also smooth. $\boxed{//}$

<u>Cor</u>: A v.s. E carries a global frame

$\iff$ it is trivializable.

$\Uparrow$

$\exists$ global trivialization

$$\begin{tikzcd} E \arrow[r, "\cong"] \arrow[dr, "\pi"'] & M \times \mathbb{R}^k \arrow[d, "p_1"] \\ & M \end{tikzcd}$$