

October 10

Recall $T_p M = \left\{ \begin{array}{l} \text{Derivations at } p \\ C^\infty(M) \xrightarrow{\gamma} \mathbb{R} \end{array} \right\}$

In particular, a v.f. X on M is an assignment of a derivation X_p for each $p \in M$.

Given $f \in C^\infty(M)$ define

$$Xf \in C^\infty(M)$$

by $(Xf)(p) = X_p(f)$.

Prop: Let $X: M \rightarrow TM$ be a section of $\pi: TM \rightarrow M$. TFAE:

1) X is a smooth v.f.

2) $\forall f \in C^\infty(M)$, $Xf \in C^\infty(M)$

3) For each $U \subset_{\text{op}} M$, $f \in C^\infty(U)$, then Xf is smooth on U .

Pf: 1) \Rightarrow 2) Spcs X is smooth, let

$f \in C^\infty(M)$. Then in coordinates

$$Xf(x) = \sum_{i=1}^n X^i(x) \frac{\partial f}{\partial x_i}(x)$$

Since $X^i(x)$ are smooth so is $Xf(x)$.

2) \Rightarrow 3) Spcs $U \subset M$ is open, $f \in C^\infty(U)$.
For $p \in U$, let ψ be a bump function

1) $\psi \in C^\infty(M)$

2) $\forall V \subset U$ contains p and $\psi|_V \equiv 1$.

3) $\psi|_{M-U} \equiv 0$.

Then define $\tilde{f} = \psi f$, which is defined to be 0 away from U .

The $X \tilde{f}$ is smooth by assumption.

And

$$X \tilde{f} \Big|_V = X f \Big|_V .$$

$\Rightarrow X f$ is smooth on a nbd of each pt of U .

3) \Rightarrow 1). Take a possibly smaller nbd where there is a chart. On this nbd:

$$X(x^i) = \sum_j X^j \frac{\partial}{\partial x^j} = X^i .$$

$\Rightarrow X^i$ smooth for each i . □

So $X \in \text{Vect}(M)$ defines

$$\begin{aligned} X : C^\infty(M) &\longrightarrow C^\infty(M) \\ f &\longmapsto X f . \end{aligned}$$

[R a ring. A R -derivation (or just a derivation) is a linear map

$$D : R \rightarrow R$$

s.t. $D(ab) = D_a \cdot b + a \cdot D_b.$]

Prop: $\text{Vect}(M) \cong \text{Der}(C^\infty(M))$.
↑
vector space of all derivations.

Pf: Being a derivation at each pt implies

$$X(fg) = (Xf) \cdot g + f \cdot (Xg).$$

Conversely, sps $D \in \text{Der } C^\infty(M)$. We will produce a v.f. X s.t.

$$Df = Xf \quad \forall f \in C^\infty(M).$$

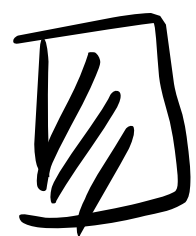
Define for each $p \in M$ $X_p \in T_p M$ by

$$X_p f = (Df)(p).$$

(Since D is derivation it follows that

X_p is a derivation at p .)

Now since Df is smooth $\Rightarrow p \mapsto X_p$
is smooth.



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Let $F: M \rightarrow N$ be smooth. Recall
that for each $p \in M$ we have

$$dF_p: T_p M \rightarrow T_p N.$$

Q: Does this define

$$"dF : \text{Vect}(M) \rightarrow \text{Vect}(N)"?$$

In general, no. For example, if F is not surjective then we do not know how to define " $dF(x)$ " as a v.f. on N . Similar issue if F is not injective.

Dfn: $X \in \text{Vect}(M)$ and $Y \in \text{Vect}(N)$ are F -related if

$$dF_p(X_p) = Y_{F(p)}$$

for all $p \in M$.

Prop: $X \underset{F}{\sim} Y$ iff for any smooth fn

f defined on an open subset of N :

$$X(f \circ F) = Yf \circ F.$$

Pf: Let f be defined on nbhd of

$F(p) \in N$. Then

$$X(f \circ F)(p) = X_p(f \circ F) \\ = dF_p(X_p)f$$

and

$$(Yf \circ F)(p) = (Yf)_{F(p)} = Y_{F(p)}f \quad \square$$

Ex: Let $F: \mathbb{R} \rightarrow \mathbb{R}^2$

$$t \mapsto (\cos t, \sin t)$$

$$X = \frac{d}{dt} \in \text{Vect}(\mathbb{R})$$

$$Y = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \in \text{Vect}(\mathbb{R}^2)$$

Then $X \underset{F}{\sim} Y$.



Prop: If $F: M \rightarrow N$ is a diffeomorphism,

and $X \in \text{Vect}(M)$, then $\exists! Y \in \text{Vect}(N)$

s.t. $X \overset{F}{\sim} Y$. In other words

$F_* = "dF" : \text{Vect}(M) \xrightarrow{\cong} \text{Vect}(N)$

is defined and is an isomorphism.