October 10

Recall $T_pM = \left\{ \begin{array}{c} \text{Derivations at } p \\ C^\infty(M) \xrightarrow{v} \mathbb{R} \end{array} \right\}$

In particular, a v.f. X on M is an assignment of a derivation X_p for each $p \in M$.

Given $f \in C^\infty(M)$ define

$$Xf \in C^\infty(M)$$

by $$(Xf)(p) = X_p(f).$$

<u>Prop</u>: Let $X : M \to TM$ be a section of $\pi : TM \to M$. TFAE:

1) X is a smooth v.f.
2) $\forall f \in C^\infty(M)$, $Xf \in C^\infty(M)$
3) For each $U \subseteq_{op} M$, $f \in C^\infty(U)$, then Xf is smooth on U.

Pf: 1) $\Rightarrow$ 2). Sps X is smooth, let $f \in C^\infty(M)$. Then in coordinates

$$Xf(x) = \sum_{i=1}^{n} X^i(x) \frac{\partial f}{\partial x_i}(x).$$

Since $X^i(x)$ are smooth so is $Xf(x)$.

2) $\Rightarrow$ 3) Sps $U \subset M$ is open, $f \in C^\infty(U)$. For $p \in M$, let ψ be a bump function

1) $\psi \in C^\infty(M)$

2) $V \subset U$ contains p and $\psi\big|_V \equiv 1$.

3) $\psi\big|_{M-U} \equiv 0$.

Then define $\tilde{f} = \psi f$, which is defined to be 0 away from U.

Then $X\tilde{f}$ is smooth by assumption.
And

$$X\tilde{f}\Big|_V = Xf\Big|_V .$$

$\Rightarrow$ Xf is smooth on a nbd of each pt of U.

3) $\Rightarrow$ i). Take a possibly smaller nbd where there is a chart. On this nbd:

$$X(x^i) = \sum_j X^j \frac{\partial}{\partial x^j} = X^i .$$

$\Rightarrow$ X^i smooth for each i. $\blacksquare$

So $X \in \mathrm{Vect}(M)$ defines

$$X : C^\infty(M) \longrightarrow C^\infty(M)$$
$$f \longmapsto Xf .$$

[R a ring. A R-derivation (or just a derivation) is a lin map

$$D : R \longrightarrow R$$

s.t. $D(ab) = Da \cdot b + a \cdot Db$.]

<u>Prop</u>: $\mathrm{Vect}(M) \cong \underset{\substack{\uparrow \\ \text{vector space of all derivations}}}{\mathrm{Der}}(C^\infty(M))$.

<u>Pf</u>: Being a derivation at each pt implies

$$X(fg) = (Xf) \cdot g + f \cdot (Xg).$$

Conversely, sps $D \in \mathrm{Der}\, C^\infty(M)$. We will produce a v.f. X s.t.

$$Df = Xf \quad \forall f \in C^\infty(M).$$

Define for each $p \in M$ $X_p \in T_pM$ by

$$X_p f = (\mathcal{D}f)(p).$$

(Since $\mathcal{D}$ is derivation it follows that X_p is a derivation at p.)

Now since $\mathcal{D}f$ is smooth $\Rightarrow$ $p \mapsto X_p$ is smooth.

$\blacksquare$

Let $F: M \to N$ be smooth. Recall that for each $p \in M$ we have

$$dF_p : T_pM \longrightarrow T_pN.$$

Q: Does this define

$$\text{“}dF : \mathrm{Vect}(M) \to \mathrm{Vect}(N)\text{”}?$$

In general, no. For example, if F is not surjective then we do not know how to define "$dF(X)$" as a v.f. on N. Similar issue if F is not injective.

Dfn: $X \in \mathrm{Vect}(M)$ and $Y \in \mathrm{Vect}(N)$ are F-related if

$$dF_p(X_p) = Y_{F(p)}$$

for all $p \in M$.

Prop: $X \underset{F}{\sim} Y$ iff for any smooth fn f defind on an open subset of N:

$$X(f \circ F) = Yf \circ F.$$

Pf: Let f be defnd on nbd of $F(p) \in N$. Then

$$X(f\circ F)(p) = X_p(f\circ F)$$

$$= dF_p(X_p)\, f.$$

$$\|$$

and

$$(Yf\circ F)(p) = (Yf)_{F(p)} = Y_{F(p)} f. \quad \blacksquare$$

Ex: Let $F : \mathbb{R} \longrightarrow \mathbb{R}^2$

$$t \longmapsto (\cos t, \sin t)$$

$$X = \frac{d}{dt} \in \mathrm{Vect}(\mathbb{R})$$

$$Y = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \in \mathrm{Vect}(\mathbb{R}^2)$$

Then $X \underset{F}{\sim} Y$. $\boxed{/\!/}$

<u>Prop</u>: If $F : M \to N$ is a diffeomorphism, and $X \in \mathrm{Vect}(M)$, then $\exists!\ Y \in \mathrm{Vect}(N)$ s.t. $X \overset{F}{\sim} Y$. In other words

$$F_* = \text{"} dF \text{"} : \mathrm{Vect}(M) \xrightarrow[\cong]{\checkmark} \mathrm{Vect}(N)$$

is defined and is an isomorphism.