October 20

G = Lie group.

$$\mathrm{Lie}(G) = \left\{ \begin{matrix} \text{left invariant} \\ \text{vector fields} \\ \text{on } G \end{matrix} \right\} \subset \mathrm{Vect}(G).$$

Thm: There is a vector space isomorphism

$$\varepsilon : \mathrm{Lie}(G) \xrightarrow{\cong} T_e G.$$

Pf: The map is $\varepsilon(X) = X_e$. Given $v \in T_e G$, define $\theta(v) : G \to TG$

$$\theta(v)_g = (dL_g)_e v.$$

Clearly v is a section of TG. We need to show that it is smooth. We will do this at the end.

So $\Theta : T_e G \longrightarrow \mathrm{Vect}(G)$. But for $v \in T_e G$ we have

$$L_{g*}\Theta(v)_h = (dL_g)_h \circ (dL_h)_e v$$

$$= d(L_g \circ L_h)_e v$$

$$= (dL_{gh})_e v$$

$$= \Theta(v)_{gh}.$$

$\Rightarrow \Theta(v) \in \mathrm{Lie}(G)$. Θ is inverse to ε since

$$\Theta \circ \varepsilon(X)_g = \Theta(X_e)_g = X_g.$$

$$\varepsilon \circ \Theta(v) = \varepsilon\big((dL_g)_e v\big) = v.$$

∎

Cor: Every (not nec. smooth) left invariant section of TG is smooth.

Pf: $v = X_e \Rightarrow X = \theta(v)$ which is smooth. ∎

Cor: Every Lie group admits a global frame.

Pf: Any basis for T_eG does the trick. ∎

Ex: · $\mathrm{Lie}(\mathbb{R}^n, +) \cong \mathbb{R}^n$ where the Lie bracket is trivial.

Here left translation

$$L_a(b) = a + b.$$

$$\Rightarrow \quad dL_a = \mathbb{1} \quad \text{in std coordinates.}$$

So, $X = X^i \frac{\partial}{\partial x^i}$ is left invariant

iff $X^i \equiv$ constants. $\Rightarrow$

$$\mathrm{Lie}(\mathbb{R}^n, +) \cong \mathrm{span}\left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right\}.$$

- Similarly $\mathrm{Lie}(S^1) \cong \mathbb{R} = \mathrm{span}\left\{ \frac{\partial}{\partial \theta} \right\}$.

These are all abelian.

<u>Prop</u>: There is a Lie alg isomorphism

$$\mathrm{Lie}(GL(n, \mathbb{R})) \xrightarrow{\cong} \mathfrak{gl}(n, \mathbb{R}).$$

<u>Pf</u>: First we spell out the isomorphism of vector spaces

$$T_{\mathbb{1}} GL(n, \mathbb{R}) \cong \mathfrak{gl}(n, \mathbb{R}).$$

Choose coordinate $\{x^i_j\}$ for $\mathbb{R}^{n^2} \cong \mathrm{Mat}_n$.

Basis:

$$\left\{ \frac{\partial}{\partial x^i_j}\Big|_{\mathbb{1}} \right\} \subset \mathrm{Mat}_n \cong T_{\mathbb{1}} GL(n,\mathbb{R}).$$

Then

$$\begin{array}{ccc} A^i_j \dfrac{\partial}{\partial x^i_j}\Big|_{\mathbb{1}} & \longleftrightarrow & (A^i_j) \\ \cap & & \cap \\ T_{\mathbb{1}} GL(n,\mathbb{R}) & & \mathfrak{gl}(n,\mathbb{R}) \\ & & \| \\ & & \mathrm{Mat}_n \end{array}$$

Let $\mathfrak{g} = \mathrm{Lie}(G)$. Given $A = (A^i_j) \in$

$\mathfrak{gl}(n,\mathbb{R})$ we have the vf.

$$\theta(A)\Big|_X = (dL_X)_{\mathbb{1}} A$$
$$= (dL_X)_{\mathbb{1}} \left(A^i_j \frac{\partial}{\partial x^i_j}\Big|_{\mathbb{1}} \right)$$

Since $\underline{L_X}$ is the restriction of a linear map, we have

$$dL_X = L_X .$$

Thus

$$\theta(A)_X = X \cdot A = X^i_j A^j_k \frac{\partial}{\partial x^i_k}\Big|_{\mathbb{1}}$$

Now

$$[\theta(A), \theta(B)] = \theta(A \cdot B - B \cdot A).$$

$\boxed{/\!/\!/}$

<u>Cor</u>: If V is a f.d. vector space then $\mathrm{Lie}(GL(V)) \cong \mathfrak{gl}(V)$.