October 25

Examples of flows.

Ex: Let

$$\theta : \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$(t; x, y) \longmapsto (x+t, y).$$

For $t > 0$, $\theta_t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ translates the plane to the right. This is the global flow corresponding to the vector field

$$V = \frac{\partial}{\partial x}$$

in the sense that for any $(a,b) \in \mathbb{R}^2$, the curve

$$\theta_{\cdot}(a,b) : \mathbb{R} \to \mathbb{R}^2$$

is an integral curve for V.

Ex: Consider the flow

$$\theta : \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(t\,;\, x, y) \longmapsto$$
$$(x\cos t - y\sin t,\ x\sin t + y\cos t).$$

This is the flow corresponding to the v.f.:

$$V = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}.$$

These examples were special since they were "global" flows. Not every v.f. is the infinitesimal generator of a global flow.

Ex: Let $V = x^2 \frac{\partial}{\partial x} \in \mathrm{Vect}(\mathbb{R}^2)$.

If $\gamma(t) = (x(t), y(t))$ is an integral curve then

$$x'(t) = x(t)^2.$$

$$\Rightarrow \quad x(t) = \frac{1}{1-t} + a$$

$$y(t) = b$$

for constants (a, b). In particular, the integral curve passing through $(1, 0)$ at $t = 0$ is

$$\gamma(t) = \left(\frac{1}{1-t}, 0 \right).$$

Cannot be defined for all t!